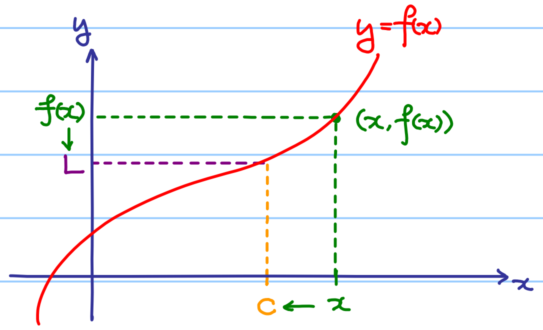
§ 5 A Glimpse of Real Analysis

"Definition" : (Limit of a function)

$L \in \mathbb{R}$ is said to be a limit of f at c if when x is getting closer and closer to c, but NOT equal to c, $f(x)$ is getting closer and closer to L.

If L is a limit of f at c, we denote it by $\lim_{x \to c} f(x) = L$.



+ Note : a little bit misleading !

$f(c)$ may NOT equal to L, even it may be undefined !

Note :

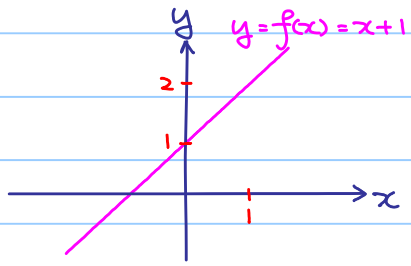
1) Limit of a function at a point must be a real number.

2) We do NOT care how f behaves at the point c.

Example 5.1

Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = x+1$.

Find $\lim_{x \to 1} f(x)$.



+

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

We write $\lim_{x \to 1} f(x) = 2$.

Remarks :

1) + The table only gives an intuitive idea, but NOT a rigorous proof !

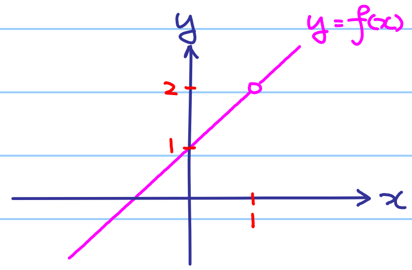
2) Do NOT regard finding limit as putting $x=1$ into $f(x)$ and getting $f(1) = 2$!

Example 5.2

Let $f: \mathbb{R}\setminus\{1\} \to \mathbb{R}$ be a function defined by $f(x) = \frac{x^2-1}{x-1}$

We can rewrite f as the following:

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	undefined	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

(But, we do NOT care what happens when $x=1$!)

We still have $\lim_{x\to 1} f(x) = 2$.

Compare with the previous example !

Remark: 1 is not in the domain of f, but we still talk about $\lim_{x\to 1} f(x)$.

Definition 5.1

Let $A \subseteq \mathbb{R}$ A point $c \in \mathbb{R}$ is a cluster point of A if

for every $\delta > 0$, there exists at least one point $x \in A\setminus\{c\}$ such that $|x-c| < \delta$

$(\forall \delta > 0)(\exists x \in A\setminus\{c\})(|x-c| < \delta)$

Rewrite in another way:

Define $V_\delta(c) = (c-\delta, c+\delta)$

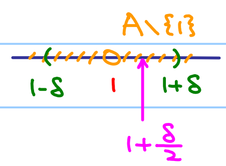
c is a cluster point of A if $(\forall \delta > 0)(V_\delta(c) \cap A\setminus\{c\} \neq \phi)$

Example 5.3

Show that $c=1$ is a cluster point of $A = \mathbb{R}\setminus\{1\}$.

Let $\delta > 0$, note that $1+\frac{\delta}{2} \in V_\delta(1) \cap (A\setminus\{1\})$

$\therefore V_\delta(1) \cap (A\setminus\{1\}) \neq 0$



Remark: Any real number is a cluster point of A

Examples 5.4

1) If $A=(0,1)$, then any point in $[0,1]$ is a cluster point of A.

2) If A is a finite subset of $\mathbb{R}$, then A has no cluster point.

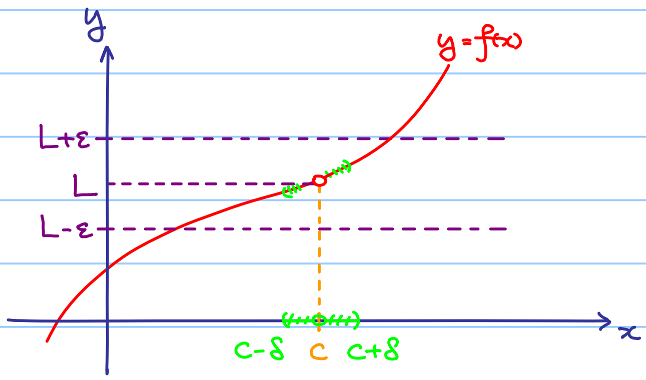
Definition 5.2

Let $A \subseteq \mathbb{R}$ and let c be a cluster point of A. For $f: A \to \mathbb{R}$, $L \in \mathbb{R}$ is to be a limit of f at c if

for all $\varepsilon>0$, there exists $\delta>0$ such that if $x \in A$ and $0<|x-c|<\delta$, then $|f(x)-L|<\varepsilon$

$(\forall \varepsilon>0)(\exists \delta>0)(\forall x \in V_\delta(c) \cap A \setminus \{c\})(|f(x)-L|<\varepsilon)$

Geometrical meaning:



Remarks:

1) If $A=\{0,1\}$, $f: A \to \mathbb{R}$,

then we cannot define $\lim_{x\to 0} f(x)$ as 0 is NOT a cluster point of A.

2) When we say $\lim_{x\to c} f(x)$, the domain of f is assumed to be the maximum domain that f can be defined.

Example 5.4

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=2x+1$. Show that $\lim_{x\to 1} f(x)=3$.

Let $\varepsilon>0$.

How to find $\delta>0$ such that when $0<|x-1|<\delta$, we have $|f(x)-3|<\varepsilon$?

$$|f(x)-3| \overset{?}{<} \varepsilon$$

$$|2x-2| \overset{?}{<} \varepsilon$$

$$|x-1| \overset{?}{<} \frac{\varepsilon}{2}$$

Take $\delta=\frac{\varepsilon}{2}>0$, then for all x with $0<|x-1|<\delta$

we have $|x-1|<\delta=\frac{\varepsilon}{2}$

$$|2x-2|<\varepsilon$$

$$|(2x+1)-3|<\varepsilon$$

$$|f(x)-3|<\varepsilon$$

$\therefore \lim_{x\to 1} f(x)=3$

Example 5.2 (Cont.)

Let $f:\mathbb{R}\setminus\{1\}\to\mathbb{R}$ be a function defined by $f(x)=\frac{x^2-1}{x-1}$ Show that $\lim_{x\to 1} f(x)=2$.

Let $\varepsilon>0$.

How to find $\delta>0$ such that when $0<|x-1|<\delta$, we have $|f(x)-2|<\varepsilon$?

$$\left|\frac{x^2-1}{x-1}-2\right| \overset{?}{<} \varepsilon$$

$$|(x+1)-2| \overset{?}{<} \varepsilon \qquad (\text{Note } x\neq 1)$$

$$|x-1| \overset{?}{<} \varepsilon$$

Take $\delta=\varepsilon>0$, then for all x with $0<|x-1|<\delta$

we have $|x-1|<\delta$

$$|(x+1)-2|<\varepsilon$$

$$\left|\frac{x^2-1}{x-1}-2\right|<\varepsilon \qquad \text{Note: } 0<|x-1| \Rightarrow x-1\neq 0$$

$$|f(x)-2|<\varepsilon$$

$\therefore \lim_{x\to 1} f(x)=2$

Example 5.5

Let $f:\mathbb{R}\to\mathbb{R}$ be a function defined by $f(x)=x^2$. Show that $\lim_{x\to 2} f(x)=4$.

Let $\varepsilon>0$.

How to find $\delta>0$ such that when $0<|x-2|<\delta$, we have $|f(x)-4|<\varepsilon$?

$|x^2-4| \overset{?}{<} \varepsilon$

$|(x-2)(x+2)| \overset{?}{<} \varepsilon$

require $\delta<1$

$|x-2|<\delta \Rightarrow 2-\delta<x<2+\delta \Rightarrow 4<4-\delta<x+2<4+\delta<5$

Then $|x+2|<5$

$|x-2|<\delta=\frac{\varepsilon}{5}$

Take $\delta=\min\{1,\frac{\varepsilon}{5}\}>0$, then for all x with $0<|x-2|<\delta$

we have $|x-2|<\delta\le\frac{\varepsilon}{5}$ and $|x-2|<\delta\le 1$

$-1\le x-2\le 1$

$-5\le 3\le x+2\le 5$

$|x+2|\le 5$

Then $|f(x)-4|=|x^2-4|=|x-2||x+2|\le\frac{\varepsilon}{5}\cdot 5=\varepsilon$

$\therefore \lim_{x\to 2} f(x)=4$.

Exercise 5.1

1) Let $b,c\in\mathbb{R}$. Prove that

a) $\lim_{x\to c} x=c$

b) $\lim_{x\to c} b=b$

c) $\lim_{x\to c} x^2=c^2$

2) Show that $\lim_{x\to 0}|x|=0$

Proposition 5.1 (Uniqueness of limits)

Let $f: A \to \mathbb{R}$ and let c be a cluster point of A.

If $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = L'$, then $L = L'$.

proof:

Suppose $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} f(x) = L'$.

Given $\varepsilon > 0$, there exists $\delta_1, \delta_2 > 0$ such that

$|f(x) - L| < \frac{\varepsilon}{2}$ for all $x \in A$ with $0 < |x - c| < \delta_1$

$|f(x) - L'| < \frac{\varepsilon}{2}$ for all $x \in A$ with $0 < |x - c| < \delta_2$

Take $\delta = \min\{\delta_1, \delta_2\} > 0$.

Since c is a cluster point of A, $A \cap V_\delta(c)$ is nonempty.

Pick $x_0 \in A \cap V_\delta(c)$, we have

$$\begin{aligned} |L - L'| &= |L - f(x_0) + f(x_0) - L'| \\ &\leq |f(x_0) - L| + |f(x_0) - L'| \qquad (\delta \leq \delta_1, \delta_2 \Rightarrow |x_0 - c| < \delta_1, \delta_2) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Since ε can be arbitrarily small, $L - L' = 0$ i.e. $L = L'$.

Definition 5.3

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A. We say f is bounded on a neighborhood of c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that $|f(x)| < M$ for all $x \in A \cap V_\delta(c)$

Proposition 5.2

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $c \in \mathbb{R}$ be a cluster point of A and $\lim_{x\to c} f(x) = L$,

then f is bounded on neighborhood of c

proof:

Choose $\varepsilon = 1$,

there exists $\delta > 0$ such that

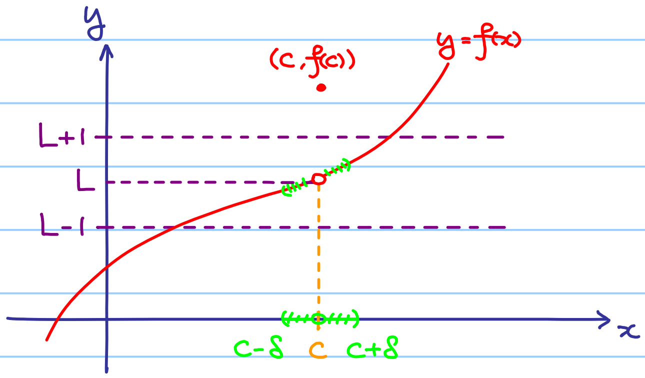
$|f(x) - L| < \varepsilon = 1 \quad \forall x \in A \setminus \{c\} \cap V_\delta(c)$

i.e. $L-1 < f(x) < L+1$

$\Rightarrow |f(x)| < |L| + 1$

Take $M = \max\{f(c), |L|+1\}$ (if $f(c)$ is defined)

or $|L|+1$ (if $f(c)$ is NOT defined)



Proposition 5.3 (Algebraic properties)

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A

If $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, then

(1) $\lim_{x\to c} f(x) \pm g(x) = L \pm M$

(2) $\lim_{x\to c} f(x)g(x) = LM$

(3) If $g(x) \neq 0$ for all $x \in A$ and $M \neq 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \frac{L}{M}$

proof of (2):

Since $\lim_{x\to c} f(x) = L$, there exists $\delta' > 0$ and $Q > 0$ such that $|f(x)| < Q$ for all $x \in A \cap V_{\delta'}(c)$

Given $\varepsilon > 0$, there exists $\delta'', \delta''' > 0$ such that

$|f(x) - L| < \frac{\varepsilon}{2|M|}$ for all $x \in A \setminus \{c\} \cap V_{\delta''}(c)$ and $|g(x) - M| < \frac{\varepsilon}{2Q}$ for all $x \in A \setminus \{c\} \cap V_{\delta'''}(c)$

Take $\delta = \min\{\delta', \delta'', \delta'''\} > 0$, then for all $x \in A \setminus \{c\} \cap V_\delta(c)$, we have

$|f(x)g(x) - LM|$

$= |f(x)g(x) - f(x)M + f(x)M - LM|$

$\leq |f(x)||g(x) - M| + |f(x) - L||M|$

$< Q \cdot \frac{\varepsilon}{2Q} + \frac{\varepsilon}{2|M|} \cdot |M|$

$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

Exercise 5.2

Prove that if $P(x)$ is a polynomial, then $\lim_{x\to c} P(x) = P(c)$

Exercise 5.3

Let $A \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A.

If $a \le f(x) \le b$ for all $x \in A \setminus \{c\}$ and $\lim_{x\to c} f(x)$ exists, show that $a \le \lim_{x\to c} f(x) \le b$.

Theorem 5.1 (Sandwich Theorem)

Let $A \subseteq \mathbb{R}$, $f, g, h: A \to \mathbb{R}$ and $c \in \mathbb{R}$ be a cluster point of A.

If $f(x) \le g(x) \le h(x)$ for all $x \in A \setminus \{c\}$ and $\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = L$.

proof:

Given $\varepsilon > 0$, there exists $\delta', \delta'' > 0$ such that

$|f(x) - L| < \varepsilon$ for all $x \in A \setminus \{c\} \cap V_{\delta'}(c)$ and $|h(x) - L| < \varepsilon$ for all $x \in A \setminus \{c\} \cap V_{\delta''}(c)$

Take $\delta = \min\{\delta', \delta''\}$, then for all $x \in A \setminus \{c\} \cap V_{\delta}(c)$, we have

$-\varepsilon < f(x) - L < g(x) - L < h(x) - L < \varepsilon$

$\therefore$ $|g(x) - L| < \varepsilon$ and so $\lim_{x\to c} g(x) = L$.

Example 5.6

Show that $\lim_{x\to 0} x \sin\frac{1}{x} = 0$

$-1 \le \sin\frac{1}{x} \le 1$ for all $\mathbb{R} \setminus \{0\}$

$-|x| \le x \le |x|$

$\therefore$ $-|x| \le x \sin\frac{1}{x} \le |x|$ for all $\mathbb{R} \setminus \{0\}$

Also $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$,

by sandwich theorem, $\lim_{x\to 0} x \sin\frac{1}{x} = 0$.