THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5510 Foundation of Advanced Mathematics 2017-2018
Assignment 2 (Due date: 26 Oct, 2017)

1. Show that the function $f : [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2$ is injective.

   Furthermore, is f a surjective function? Why?

2. Let A, B and C be sets and let $g : A \to B$ and $f : B \to C$ be bijective functions.

   Show that $(f \circ g) : A \to C$ is also a bijective function.

3. (Optional) Intermediate value theorem: Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < f(b)$. If L is a real number such that $f(a) < L < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = L$

   By using intermediate value theorem, show that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is surjective.

   (You may assume f is a continuous function.)

4. By using the definition of addition of natural numbers, evaluate $2 + 3$.

   (You may assume the usual notations $1 = 0^+$, $2 = 1^+$, $3 = 2^+$ and etc.)

5. Recall the definition of multiplication of natural numbers: Let $m, n \in \mathbb{N}$,

   - $m \times 0 = 0$;
   - $m \times n^+ = m \times n + m$.

   Follow the steps below to prove various properties of multiplication of natural numbers.

   (a) $0 \times m = 0$ for all $m \in \mathbb{N}$. (Prove by induction on m)

   (b) (Existence of Identity) $1 \times m = m \times 1 = m$ for all $m \in \mathbb{N}$. (Prove by induction on m)

   (c) $m^+ \times n = m \times n + n$ for all $m, n \in \mathbb{N}$. (Prove by induction on n)

   (d) (Commutative Law of Multiplication) $m \times n = n \times m$ for all $m, n \in \mathbb{N}$. (Prove by induction on m)

   (e) (Distributive Law) $m \times (n + p) = m \times n + m \times p$ for all $m, n, p \in \mathbb{N}$. (Prove by induction on p)

   (f) (Associative Law of Multiplication) $(m \times n) \times p = m \times (n \times p)$ for all $m, n, p \in \mathbb{N}$. (Prove by induction on p)

6. Prove that for any natural number n, there is no natural number m such that $n < m < n^+$.

7. (a) Let $m, n \in \mathbb{N}$. Prove that $m < n$ if and only if $m^+ < n^+$.

   (b) Let $m, n, p \in \mathbb{N}$. Prove that $m < n$ if and only if $m + p < n + p$.

   (Hint: Using mathematical induction on p.)

8. Let $m, n, p \in \mathbb{N}$ and $p \neq 0$. Prove that $m < n$ if and only if $mp < np$.