# Solution to Assignment 7

1. Consider the linear partial differential equation of second order with two variables

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G ,$$

where $A, B, C, D, E, F$ and $G$ are given functions of $(x, y)$ in some plane region $D$. The equation is called homogeneous if $G \equiv 0$. Show that if $u_1$ and $u_2$ are solutions to this equation in the homogeneous case, $a_1u_1 + a_2u_2$ is again a solution for any $a_1$ and $a_2$. When $G \neq 0$, show that every solution can be written as $u = v + w$ where $v$ is a solution to the corresponding homogeneous equation and $w$ is a particular solution to the full equation. (Note. The same result applies to all linear PDE's of all orders and variables.)

**Solution.** A straightforward verification. It shows that all solutions to a homogeneous linear differential equation form a vector space.

2. Consider the initial-boundary value problem under the Neumann condition

$$\begin{cases} u_t = u_{xx} & \text{in } [0, \pi] \times (0, \infty) , \\ u(x, 0) = f(x) & \text{in } [0, \pi], \\ u_x(0, t) = u_x(\pi, t) = 0 , \quad t > 0, \end{cases} \tag{1}$$

(a) By extending the solution to $[-\pi, \pi]$ as an even function in $x$, use cosine series to find the solution of this problem. (This was done in class.)

(b) Use the method of separation of variables to solve the problem.

**Solution.** We do (b) only. First find all separated solutions $X(x)T(t)$. Plugging in the equation leads to the problem $X'' + \lambda X = 0, X'(0) = X'(\pi) = 0$ and $T' + \lambda T = 0$. It is routine to show that there no non-trivial solutions for $X$ when $\lambda < 0$. When $\lambda = 0$, $X$ is any non-zero constant. When $\lambda > 0$, it must be equal to $n^2, n \geq 1$ and $X$ is a non-zero multiple of $\cos nx$. Correspondingly we have $T(t) = e^{-n^2 t}$. Hence the formal solution is

$$u(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2 t} \cos nx ,$$

where $a_n$'s are determined by the expansion

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

3. Optional. Instead of separation of variables, use Fourier series to study the normalized heat equation under $u_x(0, t) = 0,\ u(\pi, t) = 0$ . Hint: You need to extend $u$ to become a $4\pi$-periodic function.

**Solution.** Extend $f$ as an odd function with respect to the axis $x = \pi$ so to get a function defined on $[0, 2\pi]$. Then extend it as an even function over $[-2\pi, 2\pi]$. Keeping doing this we obtain a $4\pi$-periodic even function which is odd with respect to $x = \pi$. As a $4\pi$-periodic function

$$u(x, t) = \frac{a_0(t)}{2} + \sum_{n=1}^{\infty} a_n(t) \cos \frac{nx}{2} .$$

The condition $u(\pi,t)=0$ implies $a_{2n}=0$, so

$$u(x,t)=\frac{a_0(t)}{2}+\sum_{n=0}^{\infty}a_{2n+1}(t)\cos\frac{(2n+1)x}{2}\ .$$

Note. Although the Fourier approach works after some tricky extension, the method of separation of variables is more straightforward. It shows that when it comes to the determination of the formal solution, separation of variables is more efficient.

4. Consider the heat equation ($l=1,\kappa=1$) under the Robin condition $u(0,t)=0,\ u_x(1,t)+u(1,t)=0$. Show that all eigenvalues of the corresponding problem are given by $\lambda_n, n\geq 0$, where $\lambda_n\in((2n-1)^2\pi^2/4, n^2\pi^2)$ either analytically or by plotting graphs. Then find a formal solution to this problem.

**Solution.** Parallel to the discussion in the lecture notes, we need to solve

$$X''=-\lambda X,\quad X(0)=0,\quad X'(1)+X(1)=0\ .$$

Here $X(x)=\alpha\cos\sqrt{\lambda}x+\beta\sin\sqrt{\lambda}x$. $X(0)=0$ implies $\alpha=0$ and $X'(1)+X(1)=0$ implies that $\lambda$ should satisfy

$$\tan\sqrt{\lambda}=-\sqrt{\lambda}.$$

By plotting taking $\sqrt{\lambda}$ as the $x$-axis, we see that there is exactly one $\sqrt{\lambda_n}$ in $((n-1/2)\pi, n\pi), n\geq 1$ .

As in the lecture notes, there is no non-zero solution to $\lambda\leq 0$.

The formal solution is equal to

$$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-\lambda_n t}\sin\mu_n x\ ,$$

where

$$f(x)\sim\sum_{n=1}^{\infty}A_n\sin\mu_n x\ .$$

5. Consider the eigenvalue problem on $[0,1]$:

$$(p(x)X')'+q(x)X=-\lambda X,\quad \alpha X'(0)+\beta X(0)=0,\quad \gamma X'(1)+\delta X(1)=0\ ,$$

where $p,q$ are nice functions and $\alpha\beta\neq 0,\ \gamma\delta\neq 0$. Show that the eigenfunctions corresponding to different eigenvalues are orthogonal on $[0,1]$.

**Solution.** Indeed, we have

$$\begin{aligned}
-\lambda_1\int_0^1 X_1X_2dx &= \int_0^1(pX_1')'X_2dx+\int_0^1 qX_1X_2dx\\
&= pX_1'X_2\Big|_0^1-\int_0^1 pX_1'X_2'dx+\int_0^1 qX_1X_2dx\\
&= pX_1'X_2\Big|_0^1-pX_1X_2'\Big|_0^1+\int_0^1(pX_2')'X_1dx+\int_0^1 qX_1X_2dx\\
&= -\lambda_2\int_0^1 X_1X_2dx\ .
\end{aligned}$$

If $\lambda_1 \neq \lambda_2$,

$$\int_0^1 X_1(x)X_2(x)dx = 0 \ .$$

Note. This problem suggests the method of separation of variables may work for equations like

$$u_t = (p(x)u_x)_x + q(x)u \ ,$$

which is more general than the heat equation.

6. Find all solutions to the first order equation $u_t = cu_x$ where $c$ is a non-zero constant of the form $X(x)T(t)$. Can you use them to solve the initial-boundary value problem for this equation under Dirichlet boundary condition?

**Solution.** For the separated solution $X(x)T(t)$ it is readily seen that $X$ satisfies $X'(x) = -\lambda X$ so $X(x)$ is a constant multiple of $e^{-\lambda x}$ where $\lambda$ is any real number. From $T' = -c\lambda T$ we get $T$ a constant multiple of $e^{-c\lambda t}$. Therefore the separated solutions are given by

$$e^{-\lambda(x+ct)}, \quad \lambda \in \mathbb{R} \ .$$

In fact, this computation inspires us to discover that given any differentiable function $f(x)$, the function $u(x,t) = f(x+ct)$ solves the equation with initial function $f$. It shows that the imposition of any boundary condition is not natural.

7. In (5), Ex 6, we solve the initial-boundary value problem for the wave equation. Show that the solution $u$ can be expressed in the following close form:

$$u(x,t) = \frac{1}{2}(f(x-ct) + f(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(y)\, dy \ .$$

**Solution.** The solution $u$ obtained in that problem is

$$u(x,t) = \sum_{n=1}^{\infty}(c_n \cos cnt + d_n \sin cnt)\sin nx \ ,$$

where $c_n$ and $d_n$ are determined by

$$f(x) \sim \sum_{n=1}^{\infty} c_n \sin nx \ ,$$

and

$$g(x) \sim \sum_{n=1}^{\infty} cnd_n \sin nx \ .$$

Using

$$\cos cnt \sin nx = \frac{1}{2}(\sin n(x+ct) + \sin n(x-ct)) \ ,$$

and

$$\sin cnt \sin nx = \frac{1}{2}(\cos n(x-ct) - \cos n(x+ct)) \ ,$$

and also

$$\int_{x-ct}^{x+ct} g(y)\, dy = \sum_{n=1}^{\infty} cd_n(\cos n(x-ct) - \cos n(x+ct)) \ ,$$

we obtain the desired closed form.

8. Verify that the general solution to the ordinary differential equation $x'' + bx' + ax = 0,\ a, b \in \mathbb{R}, b^2 \neq 4a$, is given by $x(t) = Ae^{\alpha t} + Be^{\beta t}$ where $\alpha, \beta \in \mathbb{C}$, are the roots of the quadratic equation $y^2 + by + a = 0$. Can you find the general solution when $b^2 = 4a$ ?

   **Solution.** Try solution of the form $x(t) = e^{\alpha t}$. Plug in to find that $\alpha$ must satisfy $\alpha^2 + b\alpha + a = 0$. Hence when $b^2 \neq 4a$, there are two linearly independent solutions given by $e^{\alpha t}$ and $e^{\beta t}$ where $\alpha$ and $\beta$ are two distinct roots of this quadratic equation. When $b^2 = 4a$, there is a single root $\alpha$. One can check that the other independent solution is given by $te^{\alpha t}$.

   If one prefers to have real solutions, when the roots are complex $\alpha = c + id$ and $\beta = c - id$, we choose the two independent solutions to be $e^{cx} \cos dx$ and $e^{cx} \sin dx$.

9. Consider the modified wave equation $u_{tt} = u_{xx} - 2\beta u_t,\ \beta \in (0, 1)$, under the Dirichlet condition $u(0, t) = u(\pi, t) = 0$ and initial conditions $u(x, 0) = f(x),\ u_t(x, 0) = 0$. Using Fourier series or separation of variables to show that the formal solution is given by

$$u(x,t) = e^{-\beta t} \sum_{n=1}^{\infty} B_n \left( \cos \alpha_n t + \frac{\beta}{\alpha_n} \sin \alpha_n t \right) \sin nx \ , \quad \alpha_n = \sqrt{n^2 - \beta^2} \ ,$$

   where $B_n$ is the coefficient of the sine series of $f$. Hint: You need the previous problem.

   **Solution.** Straightforward.

10. Consider the nonhomogeneous wave equation

$$u_{tt} = c^2 u_{xx} - g \ ,$$

   where $g$ is a positive constant under the Dirchlet condition $u(0, t) = u(\pi, t) = 0$ and initial conditions $u(x, 0) = u_t(x, 0) = 0$. Use separation of variables to show that a formal solution is given by

$$u(x,t) = \frac{4g}{\pi c^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos(2n-1)ct \ \sin(2n-1)x - \frac{g}{2c^2} x(\pi - x) \ .$$

   **Solution.** We set

$$w = u + \frac{g}{2c^2} x(\pi - x)$$

   so that $w_{tt} = c^2 w_{xx}$ and $w(0, t) = w(\pi, t) = 0,\ w(x, 0) = gx(\pi - x)/2c^2$ and $w_t(x, 0) = 0$. Since $w$ satisfies the wave equation and Dirichlet condition, we have

$$w(x,t) = \sum_{n=1}^{\infty} (a_n \cos nct + b_n \sin nct) \sin nx \ .$$

   Using $w_t(x, 0) = 0$ we get $b_n = 0$ for all $n$, so it remains to determine $a_n$. Here, we compute the sine series of $x(\pi - x)$ to get

$$x(\pi - x) \sim \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x \ .$$

   It follows that

$$a_{2n-1} = \frac{4g}{\pi c^2} \frac{1}{(2n-1)^3} \ ,$$

   and $a_{2n=0}$.

11. Consider

$$\begin{cases} u_t = \kappa u_{xx} + F(x,t) & (x,t) \in [0,\pi] \times [0,\infty), \ \kappa > 0, \\ u(x,0) = f(x) \ , & x \in [0,\pi], \\ u_x(0,t) - au(0,t) = \phi(t), & \\ u_x(\pi,t) + bu(\pi,t) = \psi(t), \quad a,b > 0, \quad t > 0 & \end{cases}$$

This is a nonhomogeneous heat equation with nonhomogeneous boundary conditions. Show that it has at most one solution. You may proceed formally by assuming the solutions are as regular as possible. Let $w = u_2 - u_1$ where $u_1$ and $u_2$ are solutions and show $w \equiv 0$. Suggestion: Differentiate the integral

$$\int_0^\pi w^2(x,t)dx$$

in time.

**Solution.** It suffices to show that $w \equiv 0$. By subtracting the equation it is readily seen that $w$ satisfies the homogeneous heat equation $w_t = \kappa u_{xx}$ together with the homogeneous boundary conditions $w_x(0,t) - aw(0,t) = 0, w_x(\pi,t) + bw(\pi,t) = 0$ and zero initial condition $w(x,0) = 0$. The integral

$$I(t) = \int_0^\pi w^2(x,t)dx$$

is a function of $t$ only. We have

$$\begin{aligned} I'(t) &= \int_0^\pi 2w(x,t)w_t(x,t)dx \\ &= \int_0^\pi 2\kappa w(x,t)w_{xx}(x,t)dx \\ &= 2\kappa w(x,t)w_x(x,t)\Big|_0^\pi - 2\kappa \int_0^\pi w_x^2(x,t)dx \\ &= -2\kappa \left( bw^2(\pi,t) + aw^2(0,t) + \int_0^\pi w_x^2(x,t)dx \right) \ , \end{aligned}$$

where shows that $I$ is decreasing. Since $I$ is always non-negative and $I(0) = 0$, it implies that $I(t) \equiv 0$ for all $t$. Therefore, as it is continuous, $w \equiv 0$.

12. Find the general solution to the ordinary differential equation

$$t^2 x''(t) + tx'(t) - a^2 x(t) = 0, \quad a > 0 \ .$$

Hint: Look at the equation satisfied by $y(r) = x(t), \ t = e^r$ . The answer is

$$x(t) = At^a + Bt^{-a}, \quad A, B \in \mathbb{R} \ .$$

**Solution.** A direct verification.

13. Consider

$$\begin{cases} \dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial u}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 & \text{in } \Omega, \\ u = \varphi \text{ on } \partial\Omega \ , & \end{cases}$$

where $\Omega$ is the plane domain bounded by the arcs $r = 1, r = 2$ and $\theta = 0, \theta = \pi$. The boundary data $\varphi$ is zero on $r = 1, \theta = 0, \pi$ and equal to a constant $c_0$ on $r = 2$. Roughly speaking, this domain is half of the region bounded by two concentric circles $r = 2$ and $r = 1$. Use separation of variables to solve this problem. Hint: The previous problem is needed.

**Solution.** Look for special solution $R(r)\Theta(\theta)$. The equation is given by

$$R''(r)\Theta(\theta) + \frac{1}{r}R'(r)\Theta(\theta) + \frac{1}{r^2}R(r)\Theta''(\theta) = 0 \ ,$$

which leads to

$$\Theta'' = -\lambda\Theta \ , \quad R''(r) + \frac{1}{r}R'(r) = \frac{\lambda}{r^2}R(r) \ .$$

The side condition for $\Theta$ is $\Theta(0) = \Theta(\pi) = 0$, they are given by $\sin n\theta$ for $n \geq 1$. Next the equation for $R$ becomes $r^2R'' + rR - n^2R = 0$. The previous problem tells us that the independent solutions are given by $r^n$ and $r^{-n}$. Therefore, we have

$$u(r, \theta) = \sum_{n=1} (a_n r^n + b_n r^{-n}) \sin n\theta \ .$$

We recall the sine series of the constant function

$$1 \sim \frac{4}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) \ .$$

(Here the function is equal to 1 on $[0, \pi]$ and extended as an odd function on $[-\pi, \pi]$. We have

$$c_0 = \frac{4c_0}{\pi}\left(\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right) = \frac{4c_0}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1} \ , \quad x \in [0, \pi] \ .$$

The coefficients $a_n$ and $b_n$ satisfy $a_n + b_n = 0$, and

$$a_{2n}2^{2n} + b_{2n}2^{-2n} = 0, \quad a_{2n-1}2^{2n-1} + b_{2n-1}2^{-2n+1} = \frac{4c_0}{\pi(2n-1)} \ .$$

We solve the linear system to get $a_n = b_n = 0$ when $n$ is even and

$$a_n = \frac{2^{n+2}}{n(2^{2n}-1)}\frac{c_0}{\pi} \ , \quad b_n = -a_n \ , n \text{ is odd} \ .$$

The formal solution is given by

$$u(r, \theta) = \sum_{n=1}^{\infty}(a_n r^n + b_n^{-n}) \sin n\theta \ ,$$

where $a_n, b_n$ are given above.

14. Optional. Prove the formula

$$1 + 2\sum_{n=1}^{\infty} r^n \cos nx = \frac{1 - r^2}{1 - 2r\cos x + r^2} \ , \quad r \in [0, 1).$$

**Solution.** For $z = re^{i\theta}, r \in [0, 1)$, we have $\sum_{n=0}^{\infty} z^n = 1/(1-z)$. The real part of this identity gives the desired formula.

15. Consider the two dimensional Laplace equation on the rectangle $R = \{(x,y) : x \in [0,l], y \in [0,L]\}$ satisfying the boundary conditions $u(0,y) = u(l,y) = 0, u(x,0) = f_1(x), u(x,L) = f_2(x)$ . Using separation of variables to show that the formal solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(\alpha_n \cosh\frac{n\pi y}{l} + \beta_n \sinh\frac{n\pi y}{l}\right) ,$$

where $\alpha_n$ and $\beta_n$ are determined from the relation

$$\alpha_n = a_n, \quad \alpha_n \cosh\frac{n\pi L}{l} + \beta_n \sinh\frac{n\pi L}{l} = b_n ,$$

and $a_n, b_n$ are defined via

$$f_1(x) \sim \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} , \quad f_2(y) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} .$$

**Solution.** Looking for special solution of the form $X(x)Y(y)$ leads to

$$X''(x) = -\lambda X(x), X(0) = X(l) = 0 .$$

It is known that all solutions are given by $\sin\frac{n\pi x}{l}, n \geq 1$, and $\lambda_n = \left(\frac{n\pi}{l}\right)^2$. Now, assuming the form for the solution

$$u(x,y) = \sum_{n=1}^{\infty} B_n(y) \sin\frac{n\pi x}{l} .$$

Then $B_n$ satisfies $B_n'' = \left(\frac{n\pi}{l}\right)^2 B_n$ whose independent solutions are given by $e^{\frac{n\pi}{l}y}, e^{-\frac{n\pi}{l}y}$ or $\cosh\frac{n\pi}{l}y, \sinh\frac{n\pi}{l}y$. Thus

$$u(x,y) = \sum_{n=1}^{\infty} (\alpha_n \cosh ny + \beta_n \sinh ny) \sin nx .$$

To determine $\alpha_n$ and $\beta_n$ we use $u(x,0) = f_1(x),\ f_1(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$ , and $u(x,L) = f_2(x),\ f_2(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$ to get

$$\alpha_n = a_n, \quad \alpha_n \cosh\frac{n\pi L}{l} + \beta_n \sinh\frac{n\pi L}{l} = b_n , \quad \forall n .$$

In conclusion, the formal solution is given by

$$u(x,y) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(\alpha_n \cosh\frac{n\pi y}{l} + \beta_n \sinh\frac{n\pi y}{l}\right) ,$$

where $\alpha_n$ and $\beta_n$ are defined above.