MATH5030 Notes 1

1 Fourier Series, Real and Complex Forms

Let f be a real-valued integrable function on $[-\pi, \pi]$. Its Fourier series is given by

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \ ,$$

where the Fourier coefficients are given by

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \cos ny \, dy, \quad n \geq 0,$$

and

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(y) \sin ny \, dy, \quad n \geq 1.$$

On the other hand, when f is a complex-valued integrable function on $[-\pi, \pi]$, its Fourier series is given by

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx} \ ,$$

where the Fourier coefficients are given by

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} \, dy, \quad n \in \mathbb{Z} \ .$$

As a real-valued function is also a complex-valued function (whose imaginary part vanishes), both its real and complex Fourier series are well-defined. It makes sense to relate its real and complex Fourier coefficients. In fact, using Euler's formula

$$e^{inx} = \cos nx + i \sin nx,$$

we have, for $n \geq 1$,

$$\begin{aligned} 2\pi c_n &= \int_{-\pi}^{\pi} f(y) e^{-iny} \, dy \\ &= \int_{-\pi}^{\pi} f(y)(\cos ny - i \sin ny) \, dy \\ &= \int_{-\pi}^{\pi} f(y) \cos nx \, dy - i \int_{-\pi}^{\pi} f(y) \sin ny \, dy \\ &= \pi a_n - i b_n, \quad n \geq 1 \ . \end{aligned}$$

On the other hand, for $n \geq 1$,

$$\begin{aligned} 2\pi c_{-n} &= \int_{-\pi}^{\pi} f(y) e^{iny} \, dy \\ &= \int_{-\pi}^{\pi} f(y)(\cos ny + i \sin ny) \, dy \\ &= \int_{-\pi}^{\pi} f(y) \cos ny \, dy + i \int_{-\pi}^{\pi} f(y) \sin ny dy \\ &= \pi a_n + i b_n, \quad n \geq 1 \ . \end{aligned}$$

By adding and subtracting, we obtain the relation between c_n and a_n, b_n:

$$a_n = c_n + c_{-n} \ , \quad b_n = i(c_n - c_{-n}), \quad n \geq 1 \ ,$$

and $a_0 = 2c_0$.

2 The Formula for The Partial Sums

The N-th partial sum of the Fourier series for a real-valued function f is given by

$$S_N f(x) = \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \ .$$

We will show that it can be expressed in a closed form. Indeed, recall the summation formula for the cosine function

$$\cos\theta + \cos 2\theta + \cdots + \cos N\theta = \frac{\sin\left(N + \frac{1}{2}\right)\theta - \sin\frac{1}{2}\theta}{2\sin\frac{\theta}{2}}, \quad \theta \neq 0.$$

Indeed,

$$\begin{aligned}
(S_N f)(x) &= \frac{a_0}{2} + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy + \sum_{n=1}^{N} \frac{1}{\pi}\int_{-\pi}^{\pi} f(y)(\cos ny \cos nx + \sin ny \sin ny)\,dy \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\frac{1}{2} + \sum_{n=1}^{N} \cos n(y-x)\Big) f(y)\,dy \\
&= \frac{1}{\pi}\int_{x-\pi}^{x+\pi} \Big(\frac{1}{2} + \sum_{n=1}^{N} \cos nz\Big) f(x+z)\,dz \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\frac{1}{2} + \sum_{n=1}^{N} \cos nz\Big) f(x+z)\,dz \ ,
\end{aligned}$$

where in the last step we have used the fact that the integrals over any two periods are the same. Using the summation formula above, we obtain

$$\frac{1}{2} + \sum_{n=1}^{N} \cos n\theta = \frac{\sin(N + \frac{1}{2})\theta}{2\sin\frac{\theta}{2}}.$$

Noting that by L'Hospital's rule

$$\lim_{\theta \to 0} \frac{\sin(N + \frac{1}{2})\theta}{2\sin\frac{\theta}{2}} = \frac{2N+1}{2} \ ,$$

we introduce the **Dirichlet kernel** D_N by

$$D_N(z) = \begin{cases} \dfrac{\sin\left(N + \frac{1}{2}\right)z}{2\pi \sin\frac{1}{2}z}, & z \neq 0 \\ \dfrac{2N+1}{2\pi}, & z = 0. \end{cases}$$

It is a continuous, 2π-periodic function. We have successfully expressed the partial sums of the Fourier series in the following closed form:

$$(S_N f)(x) = \int_{-\pi}^{\pi} D_N(z) f(x+z)\, dy,$$

Taking $f \equiv 1$, we have $S_N f = 1$ for all N. Hence

$$1 = \int_{-\pi}^{\pi} D_N(z)\, dz.$$

Thus we have arrived at the fundamental relation

$$(S_N f)(x) - f(x) = \int_{-\pi}^{\pi} D_N(z)(f(x+z) - f(x))\, dz.$$