Assignment 6

Hand in no 1, 3, 4, and 5 by March 28.

Consider the initial-boundary value problem for the heat equation

$$\begin{cases} u_t = u_{xx} + F(x,t) & \text{in } [0,\pi] \times (0,\infty) \ , \\ u(x,0) = f(x) & \text{in } [0,\pi], \\ u(0,t) = g_1(t) \ , u(\pi,t) = g_2(t) \ , \quad t > 0, & \end{cases} \tag{1}$$

1. Find the solution of (1) when $F \equiv 0, g_1 = 5$ and $g_2 \equiv 0$. Hint: Find φ so that $v = u - \varphi$ satisfies (1) with F, g_1, g_2 all vanish.
2. Find the solution of (1) when g_1, g_2 vanish and $F(x,t) = C$. Hint: Consider the function w satisfying $w'' + C = 0, w(0) = w(\pi) = 0$.
3. Find the solution of (1) when g_1, g_2 vanish and F is nice. Hint: Make use of the Fourier expansion of $F(x,t) = \sum F_n(t) \sin nx$.
4. Find the solution of (1) when g_1, g_2 vanish and $F(x,t) = e^{-t} \sin x$.

   Consider the initial-boundary value problem for the wave equation

$$\begin{cases} u_{tt} = c^2 u_{xx} \ , c > 0 \ , & \text{in } [0,\pi] \times (0,\infty) \ , \\ u(x,0) = f(x) \ , \quad u_t(x,0) = g(x), & \text{in } [0,\pi], \\ u(0,t) = u(\pi,t) = 0 \ , \quad t > 0, & \end{cases} \tag{2}$$

5. Let $u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin nx$ be the solution to (2).
   (a) Show that b_n satisfies the differential equation

   $$b_n''(t) + n^2 c^2 b_n(t) = 0 \ .$$

   (b) Show that the solution u is given by

   $$u(x,t) = \sum_{n=1}^{\infty} (c_n \cos nt + d_n \sin nt) \sin nx \ ,$$

   where c_n and d_n are determined by

   $$f(x) \sim \sum_{n=1}^{\infty} c_n \sin nx \ ,$$

   and

   $$g(x) \sim \sum_{n=1}^{\infty} cnd_n \sin nx \ .$$