Solution 1

p. 100: 1, 7, 8

1. Prove that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are norms. Which axioms does $\|\cdot\|_p$ fails when $p<1$?

   Solution. Consider the vector space $\mathbb{K}^N$. Let $\mathbf{x}=(x_1,\dots,x_N), \mathbf{y}=(y_1,\dots,y_N)\in\mathbb{K}^N$ and $\alpha\in\mathbb{K}$.

   We first show that $\|\mathbf{x}\|_1 := \sum_{i=1}^N |x_i|$ is a norm on $\mathbb{K}^N$.

   (i) Clearly $\|\mathbf{x}\|_1\geq 0$. Moreover $\|\mathbf{x}\|_1 = 0 \Leftrightarrow |x_i|=0\,\forall 1\leq i\leq N \Leftrightarrow x_i=0\,\forall 1\leq i\leq N \Leftrightarrow \mathbf{x}=0$.

   (ii) $\|\alpha\mathbf{x}\|_1 = \sum_{i=1}^N|\alpha x_i| = |\alpha|\sum_{i=1}^N|x_i| = |\alpha|\|\mathbf{x}\|_1$.

   (iii) $\|\mathbf{x}+\mathbf{y}\|_1 = \sum_{i=1}^N|x_i+y_i| \leq \sum_{i=1}^N|x_i| + \sum_{i=1}^N|y_i| = \|\mathbf{x}\|_1+\|\mathbf{y}\|_1$.

   Hence $\|\cdot\|_1$ is a norm on $\mathbb{K}^N$.

   Next we show that $\|\mathbf{x}\|_\infty := \max_{1\leq i\leq N}|x_i|$ is a norm on $\mathbb{K}^N$.

   (i) Clearly $\|\mathbf{x}\|_\infty\geq 0$. Moreover $\|\mathbf{x}\|_\infty = 0 \Leftrightarrow |x_i|=0\,\forall 1\leq i\leq N \Leftrightarrow x_i=0\,\forall 1\leq i\leq N \Leftrightarrow \mathbf{x}=0$.

   (ii) $\|\alpha\mathbf{x}\|_\infty = \max_{1\leq i\leq N}|\alpha x_i| = |\alpha|\max_{1\leq i\leq N}|x_i| = |\alpha|\|\mathbf{x}\|_\infty$.

   (iii) $\|\mathbf{x}+\mathbf{y}\|_\infty = \max_{1\leq i\leq N}|x_i+y_i| \leq \max_{1\leq i\leq N}(|x_i|+|y_i|) \leq \max_{1\leq i\leq N}|x_i| + \max_{1\leq j\leq N}|y_j| = \|\mathbf{x}\|_\infty+\|\mathbf{y}\|_\infty$.

   Hence $\|\cdot\|_\infty$ is a norm on $\mathbb{K}^N$.

   We now show that $\|\cdot\|_p$ fails the triangle inequality, hence is not a norm, when $p<1$. For example, consider $\mathbf{x}=(1/2,0), \mathbf{y}=(0,1/2)\in\mathbb{R}^2$. Then

   $$\|\mathbf{x}+\mathbf{y}\|_p = \left(\left(\frac{1}{2}\right)^p + \left(\frac{1}{2}\right)^p\right)^{1/p} = 2^{\frac{1}{p}-1}$$

   while

   $$\|\mathbf{x}\|_p+\|\mathbf{y}\|_p = \left(\left(\frac{1}{2}\right)^p\right)^{1/p} + \left(\left(\frac{1}{2}\right)^p\right)^{1/p} = 2.$$

   Now the triangle inequality fails since $2^{\frac{1}{p}-1}>2$ whenever $p<1$. ◀

7. The norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are all equivalent on $\mathbb{R}^N$ since (prove!)

   $$\|\cdot\|_\infty \leq \|\cdot\|_2 \leq \|\cdot\|_1 \leq N\|\cdot\|_\infty.$$

   But they are not equivalent for sequences or functions! Find sequences of functions that converge in $L^1[0,1]$ but not in $L^\infty[0,1]$, or vice-versa. Can sequences converge in ℓ^1 but not in ℓ^∞?

Solution. (a) Let $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$. Using the identity $\left(\sum_{i=1}^N a_i\right)^2 = \sum_{i=1}^N a_i^2 + 2\sum_{i<j} a_i a_j$, it is easy to see that

$$\sum_{i=1}^N |x_i|^2 \le \left(\sum_{i=1}^N |x_i|\right)^2.$$

Moreover,

$$\max_{1\le i\le N} |x_i|^2 \le \sum_{i=1}^N |x_i|^2 \quad \text{and} \quad \sum_{i=1}^N |x_i| \le \sum_{i=1}^N \max_{1\le i\le N} |x_i| = N \max_{1\le i\le N} |x_i|.$$

Therefore,

$$\|\mathbf{x}\|_\infty \le \|\mathbf{x}\|_2 \le \|\mathbf{x}\|_1 \le N\|\mathbf{x}\|_\infty.$$

(b) Recall that $L^1[0,1] := \{f : [0,1] \to \mathbb{C} : \int_0^1 |f(x)|dx < \infty\}$ with norm defined by

$$\|f\|_{L^1} := \int_0^1 |f(x)|dx < \infty,$$

while $L^\infty[0,1] := \{f : [0,1] \to \mathbb{C} : f \text{ is measurable AND } \exists c\, |f(x)| \le c \text{ a.e. } x\}$ with norm defined by

$$\|f\|_{L^\infty} := \sup_{x \text{ a.e.}} |f(x)|.$$

On one hand, one can easily see that

$$\|f\|_{L^1} = \int_0^1 |f(x)|dx \le \int_0^1 \|f\|_{L^\infty} dx = \|f\|_{L^\infty},$$

so that convergence in $L^\infty[0,1]$ implies convergence in $L^1[0,1]$.
On the other hand, consider the sequence $f_n(x) := \chi_{[0,\frac{1}{n}]}$ for $n \ge 1$. Then

$$\|f_n\|_{L^1} = \int_0^{\frac{1}{n}} dx = \frac{1}{n} \to 0 \text{ as } n \to \infty,$$

while

$$\|f_n\|_{L^\infty} = \sup_{x \text{ a.e.}} |\chi_{[0,\frac{1}{n}]}| = 1.$$

Hence $(f_n)_{n\ge1}$ converges to 0 in $L^1[0,1]$ but does not converge in $L^\infty[0,1]$.

(c) Suppose a sequence $(\mathbf{x}_n)$ converges to $\mathbf{x}$ in ℓ^1. Write

$$\mathbf{x}_n = (x_1^n, x_2^n, x_3^n, \dots) \quad \text{and} \quad \mathbf{x} = (x_1, x_2, x_3, \dots).$$

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that

$$\|\mathbf{x}_n - \mathbf{x}\|_{\ell^1} := \sum_{k=1}^\infty |x_k^n - x_k| < \varepsilon/2 \text{ whenever } n \ge N.$$

In particular, for $n \geq N$, we have $|x_k^n - x_k| \leq \sum_{k=1}^{\infty} |x_k^n - x_k| < \varepsilon/2$ for $k = 1, 2, \ldots$, and hence

$$\|\mathbf{x}_n - \mathbf{x}\|_{\ell^\infty} := \sup_k |x_k^n - x_k| \leq \varepsilon/2 < \varepsilon.$$

Therefore, the sequence $(\mathbf{x}_n)$ must converge to $\mathbf{x}$ in ℓ^∞ also.

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8. **Minkowski semi-norm*: Let C be a convex set which is balanced, $e^{i\theta}C = C (\forall \theta \in \mathbb{R})$, and such that $\cup_{r>0} rC = X$. Then

$$|||x||| := \inf\{r > 0 : x \in rC\}$$

is a semi-norm on X.

Solution. (i) Since $\cup_{r>0} rC = X$, $|||x|||$ is well-defined and non-negative for any $x \in X$.

(ii) Let $x \in X$ and $\lambda \in \mathbb{C}$. First suppose $\lambda = 0$. Since $\cup_{r>0} rC = X$, we have $0 \in rC$ for some $r > 0$, and in fact $0 \in rC$ for all $r > 0$. Thus

$$|||0x||| = |||\mathbf{0}||| = 0 = 0|||x|||.$$

Next suppose $\lambda \neq 0$. Since C is balanced, we have

$$\lambda x \in rC \Leftrightarrow |\lambda| x \in rC \Leftrightarrow x \in \frac{r}{|\lambda|}C, \qquad \text{for all } r > 0.$$

Then

$$\{r > 0 : \lambda x \in rC\} = \{r > 0 : x \in \frac{r}{|\lambda|}C\} = |\lambda|\{s > 0 : x \in sC\},$$

and taking the infimum on both sides gives $|||\lambda x||| = |\lambda| |||x|||$.

(iii) Let $x, y \in X$. Suppose $x \in sC$ and $y \in tC$. Then

$$x + y \in sC + tC = (s+t)\left(\frac{s}{s+t}C + \frac{t}{s+t}C\right) \subset (s+t)C,$$

since C is convex. Therefore $|||x + y||| \leq s + t$. Taking the infimum over s and t gives

$$|||x + y||| \leq |||x||| + |||y|||.$$

Hence $|||\cdot|||$ is a semi-norm on X.

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p. 106: 3

3. When X, Y are Banach spaces over the same field, so is $X \times Y$ (Proposition 4.7).

Solution. Recall that $X \times Y$ is a normed space with norm given by

$$\|(x, y)\|_{X\times Y} := \|x\|_X + \|y\|_Y.$$

It suffices to show that if X and Y are complete then so is $X \times Y$.
Let $\{(x_n, y_n)\}_{n=1}^{\infty} \in X \times Y$ be a Cauchy sequence Then

$$\|(x_n, y_n) - (x_m, y_m)\|_{X\times Y} = \|x_n - x_m\|_X + \|y_n - y_m\|_Y \geq \|x_n - x_m\|_X.$$

Since the left-hand sequence converges to 0 as $n, m \to \infty$, we get $\|x_n - x_m\|_X \to 0$, so that the sequence $\{x_n\}_{n=1}^{\infty}$ is Cauchy in the complete space X. It therefore converges to some point x in X. By similar reasoning, $y_n \to y \in Y$. Consequently,

$$\|(x_n, y_n) - (x, y)\|_{X\times Y} = \|x_n - x\|_X + \|y_n - y\|_Y \to 0 \quad \text{as } n \to \infty,$$

which is equivalent to $(x_n, y_n) \to (x, y)$ in $X \times Y$. Thus $X \times Y$ is complete, hence a Banach space. ◀