MATH 4010

Tutorial 2

Eg 1. Let S be the set of all sequences of complex numbers.

Set $d(x,y)=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\xi_i-\eta_i|}{1+|\xi_i-\eta_i|}$, where $x=\{\xi_i\}_{i=1}^{\infty}$, $y=\{\eta_i\}_{i=1}^{\infty}$

Show that (S,d) is a complete metric space.

<u>Pf</u>: Step 1: (S,d) is a metric space.

(i) It is clear that d is nonegative and symmetric, and $d(x,y)$ iff $x=y$.

(ii) Now, it sufficies to show d satisfies the triangle ineq.

Indeed, note that $f(t)=\frac{t}{1+t}=1-\frac{1}{1+t}\nearrow$ on $[0,\infty)$

So, $f(|a+b|)\leq f(|a|+|b|)$, i.e.

$$\frac{|a+b|}{1+|a+b|}\leq\frac{|a|+|b|}{1+|a|+|b|}\leq\frac{|a|}{1+|a|+|b|}+\frac{|b|}{1+|a|+|b|}$$

$$\leq\frac{|a|}{1+|a|}+\frac{|b|}{1+|b|}$$

Then, $\forall\ x=\{\xi_i\},\ y=\{\eta_i\},\ z=\{\zeta_i\}$

by choosing $a=\xi_i-\zeta_i$, $b=\zeta_i-\eta_i$, we have

$$\frac{|\xi_i-\eta_i|}{1+|\xi_i-\eta_i|}\leq\frac{|\xi_i-\zeta_i|}{1+|\xi_i-\zeta_i|}+\frac{|\zeta_i-\eta_i|}{1+|\zeta_i-\eta_i|}$$

$$\Rightarrow\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\xi_i-\eta_i|}{1+|\xi_i-\eta_i|}\leq\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\xi_i-\zeta_i|}{1+|\xi_i-\zeta_i|}+\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|\zeta_i-\eta_i|}{1+|\zeta_i-\eta_i|}$$

Therefore, $d(x,y)\leq d(x,z)+d(z,y)$.

Step 2: (S,d) is complete.

Let $\{x^{(n)}\}$ be a Cauchy sequence, i.e.

$$d(x^{(n)},x^{(m)})=\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i^{(n)}-x_i^{(m)}|}{1+|x_i^{(n)}-x_i^{(m)}|}\longrightarrow 0\quad\text{as } n,m\to+\infty.$$

Then, $\forall\, k\in\mathbb{N}$ $\forall\, 0<\varepsilon<1$ $\exists\, N\in\mathbb{Z}^+$ s.t. $\forall\, n,m>N$

$$\frac{1}{2^k}\frac{|x_k^{(n)}-x_k^{(m)}|}{1+|x_k^{(n)}-x_k^{(m)}|}\leq\sum_{i=1}^{\infty}\frac{1}{2^i}\frac{|x_i^{(n)}-x_i^{(m)}|}{1+|x_i^{(n)}-x_i^{(m)}|}<\frac{\varepsilon}{2^{k+1}}.$$

$$\Rightarrow |x_k^{(n)}-x_k^{(m)}|<\frac{\frac{\varepsilon}{2}}{1-\frac{\varepsilon}{2}}<\varepsilon.$$

So, for any fixed k, $\{x_k^{(n)}\}$ is a Cauchy sequence in $\mathbb{C}$. By the completness of $\mathbb{C}$, there exist x_k such that $x_k^{(n)} \to x_k$ as $n \to \infty$.

Denote by $x = \{x_1, x_2, \cdots, x_k, \cdots\}$. We claim that $\{x^{(n)}\}$ converges to x, i.e. $d(x^{(n)}, x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k^{(n)} - x_k|}{1+|x_k^{(n)} - x_k|} \to 0$ as $n \to \infty$.

Indeed, $\forall \varepsilon > 0$, $\exists\ n_0 = 1 - \log_2 \varepsilon$

$$\sum_{k=n_0+1}^{\infty} \frac{1}{2^k} \frac{|x_k^{(n)} - x_k|}{1+|x_k^{(n)} - x_k|} \leq \sum_{k=N_0+1}^{\infty} \frac{1}{2^k} = \frac{1}{2^{n_0}} < \frac{\varepsilon}{2}.$$

For any $k \leq n_0$, since $\{x_k^{(n)}\}$ converge to x_k,

$$\exists\ N_k \text{ s.t. } |x_k^{(n)} - x_k| < \frac{\varepsilon}{2},\ \forall n > N_k.$$

Chosing $N = \max\{N_1, N_2, \cdots, N_{n_0}\}$, then, $\forall n > N$

$$\sum_{k=1}^{n_0} \frac{1}{2^k} \frac{|x_k^{(n)} - x_k|}{1+|x_k^{(n)} - x_k|} < \sum_{k=1}^{n_0} \frac{1}{2^k} \cdot \frac{\varepsilon}{2} < \frac{\varepsilon}{2}.$$

Therefore, $\sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k^{(n)} - x_k|}{1+|x_k^{(n)} - x_k|} < \varepsilon$. #

We do this because we can't have a uniform N for infinite many sequence $\{x_k^{(n)}\}_{n=1}^{\infty}$

Contradiction:

$x^{(n)} = \begin{cases} 0 & k \neq n \\ n & k = n \end{cases}$

$x \equiv 0$

Eg 2. Let F be the set of all sequences of real number with only finite nonzero elements. Define d by

$$d(x,y) = \sup_{k \geq 1} |\xi_k - \eta_k|,\ \forall x = \{\xi_k\},\ y = \{\eta_k\} \in F.$$

Show that (F, d) is not complete.

What is the completion of (F, d)?

Solu: ① To show (F,d) is not complete, it sufficies to construct a Cauchy sequence in F but not converge associated with metric d.

For example, we can construct as follows.

Set $x^{(n)} = \{1, \frac{1}{2}, \cdots, \frac{1}{n}, 0, 0, \cdots, 0, \cdots\}$.

It is clear that $x^{(n)} \in F$, $\forall n \in \mathbb{N}$. Moreover, w.l.o.g. $\forall m > n$

$$d(x^{(n)}, x^{(m)}) = \frac{1}{n+1} \to 0 \text{ as } n \to \infty$$

Since $x^{(n)} - x^{(m)} = \{0, \cdots, 0, \frac{1}{n+1}, \cdots, \frac{1}{m}, 0, \cdots 0, \cdots\}$.

That is $\{x^{(n)}\}$ is a Cauchy sequence in F.

Let $x = \{\frac{1}{k}\}_{k=1}^{\infty}$. Then $d(x^{(n)}, x) = \frac{1}{n+1} \to 0$ as $n \to \infty$.

So, $x^{(n)}$ converge to x. But it is clear that $x \notin F$.

② It is not difficult to guess that the completion of (F, d) has the property that sequences there must converge to zero, i.e. c_0 which consists of all the sequences converging to 0.

Now, we prove this claim:

The completion of (F, d) is (c_0, d).

Step 1: c_0 is complete.

Let $\{x^{(n)}\}$ be a Cauchy sequence in c_0, i.e.

$$d(x^{(n)}, x^{(m)}) = \sup_{k \geq 1} |x_k^{(n)} - x_k^{(m)}| \to 0 \quad \text{as} \quad n, m \to \infty.$$

Then, $\forall \varepsilon > 0$, $\exists N$ s.t. $n, m > N$,

$$\sup_{k \geq 1} |x_k^{(n)} - x_k^{(m)}| < \varepsilon \Rightarrow |x_k^{(n)} - x_k^{(m)}| < \varepsilon, \ \forall k \in \mathbb{N} \qquad (*)$$

So, $\{x_k^{(n)}\}$ is a Cauchy sequence in $\mathbb{C}$, which implies that $\exists x_k$ s.t. $|x_k^{(n)} - x_k| \to 0$ as $n \to \infty$.

Set $x = (x_1, \cdots, x_k, \cdots)$. Taking $m \to \infty$ gives that

$\forall \varepsilon > 0$, $\exists N$ s.t. $n > N$,

$$|x_k^{(n)} - x_k| < \varepsilon \quad \forall k \in \mathbb{N}$$

Therefore, it remains to show $x \in c_0$, i.e. $x_k \to 0$ as $k \to \infty$.

Indeed, $\forall \varepsilon > 0$, $\exists N$ s.t. $\forall n, k > N$,

$$|x_k| \leq |x_k - x_k^{(n)}| + |x_k^{(n)}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Step 2: F is dense in c_0, i.e. $\forall x \in c_0$, $\exists \{x^{(n)}\} \subset F$ s.t. $d(x^{(n)}, x) \to 0$ as $n \to \infty$.

Indeed, $\forall x = \{x_k\} \in c_0$, Define $x^{(n)} = \{x_1, x_2, \cdots, x_n, 0, \cdots, 0\}$.

It is clear that $x^{(n)} \in F$ and

$$d(x^{(n)}, x) = \sup_{k \geq n+1} |x_k| \to 0 \quad \text{as} \quad n \to \infty, \text{ since } x \in c_0.$$

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