Tutorial 6.25

Nonhomogeneous Linear System

$$\vec{x}' = P(t)\,\vec{x} + \vec{g}(t)$$

$P(t)$: $n \times n$ matrix , both $P(t)$, $\vec{g}(t)$ continuous in some interval $\alpha < t < \beta$.

$\vec{g}(t)$: $n \times 1$ vector

Question : How to find a particular solution for the system ?

There are 3 methods here :

① Diagonalization :

Consider the system $\vec{x}' = A\vec{x} + \vec{g}(t)$ $\cdots$ (✱)

Suppose $A_{n\times n}$ is a diagonalizable constant matrix, we mean that A has n linearly independent eigenvectors $\vec{\xi}^1, \vec{\xi}^2, \cdots \vec{\xi}^n$. Then A can be written as

$$D = T^{-1}AT \ , \quad \text{where } D = \begin{pmatrix} \lambda_1 & & O \\ & \ddots & \\ O & & \lambda_n \end{pmatrix},$$

$$T = \begin{pmatrix} \vec{\xi}_1 & \cdots & \vec{\xi}_n \end{pmatrix}_{n\times n}$$

Define a new dependent variably $\vec{y} \in \mathbb{R}^n$ by $\vec{x} = T\vec{y}$.

Substitute back to (✱) , we have

$T\vec{y}' = AT\vec{y} + \vec{g}(t)$, multiply T^{-1}

$\vec{y}' = (T^{-1}AT)\vec{y} + T^{-1}\vec{g}(t) \Rightarrow \boxed{\begin{array}{l} \vec{y}' = D\vec{y} + \vec{h}(t), \\ \vec{h}(t) = T^{-1}\vec{g}(t) \end{array}}$

Now we have n uncoupled equations for $y_1, \cdots, y_n$.

$y_i'(t) = r_i y_i(t) + h_i(t) \quad i = 1, \cdots, n.$ $\quad 1^{st}$ order ODE.

each y_i can be solved. $\vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ is a particular solution.

② Method of undetermined coefficients.

$$\vec{x}' = P(t)\vec{x} + \vec{g}(t).$$

Require: P is a constant matrix.

$\vec{g}$ contains only: exponentials, polynomials, $\sin mt$, $\cos mt$. or sums or products of them.

If $\vec{g}(t)$ is in form of $e^{\lambda t}$, λ is a simple root of the characteristic function, we should try $\boxed{\vec{x} = \vec{a}te^{\lambda t} + \vec{b}e^{\lambda t}}$ instead of $\vec{a}te^{\lambda t}$ only.

③ Variation of parameters. $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$.

Assume $\Psi(t)$ be the fundamental matrix corresponding to $\vec{x}' = P(t)\vec{x}$.

To construct a particular solution, we assume that $\vec{x} = \Psi(t)\vec{u}(t)$.
differentiate $\vec{x}$,

$$\Psi'(t)\vec{u}(t) + \Psi(t)\vec{u}'(t) = P(t)\Psi(t)\vec{u}(t) + \vec{g}(t)$$

Since $\Psi'(t) = P(t)\Psi(t)$, we have

$$\Psi(t)\vec{u}'(t) = \vec{g}(t)$$

i.e. $$\vec{u}'(t) = \Psi(t)^{-1}\vec{g}(t)$$

$$\vec{u}(t) = \int \Psi(t)^{-1}\vec{g}(t)\,dt + \vec{c}.$$ $\vec{c}$: arbitrary constant vector.

And the general solution is : $\vec{x}(t) = \Psi(t)\vec{c} + \Psi(t)\int_{t_1}^{t}\Psi^{-1}(s)\vec{g}(s)\,ds$,

where $t_1 \in (\alpha, \beta)$.

If we have an initial condition $\vec{x}(t_0) = x^0$,

then $\vec{c} = \Psi^{-1}(t_0)x^0$.

Exercise : $\quad \vec{x}' = \begin{pmatrix} 2 & 3 \\ -1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} e^t \\ t \end{pmatrix}$ find a particular solution.

$$\begin{vmatrix} 2-\lambda & 3 \\ -1 & -2-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \qquad \lambda_1 = 1, \quad \lambda_2 = -1.$$

$$\vec{\xi}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix}, \quad T^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\therefore \vec{x} = T\vec{y}$,

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = D\vec{y} + T^{-1}\vec{g}(t) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & -3 \end{pmatrix} \begin{pmatrix} e^t \\ t \end{pmatrix}$$

$$\begin{cases} y_1' = y_1 + \frac{1}{\sqrt{2}}(e^t + t) \\ y_2' = -y_2 - \frac{1}{\sqrt{2}}(e^t + 3t) \end{cases} \Rightarrow \begin{aligned} y_1 &= \frac{t^2}{2\sqrt{2}} e^t - \frac{t}{\sqrt{2}} - \frac{1}{\sqrt{2}} + c_1 e^t \\ y_2 &= -\frac{1}{2\sqrt{2}} e^t - \frac{3t}{\sqrt{2}} + \frac{3}{\sqrt{2}} + c_2 e^{-t} \end{aligned}$$

$c_1, c_2 \in \mathbb{R}$.

$$\vec{x} = T\vec{y} = \frac{1}{\sqrt{2}} \begin{pmatrix} -3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

$$\vec{X}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{X} + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}$$

Sol:

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = 0 \Rightarrow \quad \lambda_1 = i, \ \lambda_2 = -i.$$

$$\vec{\xi}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}.$$

Then $\begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$

homogeneous Solution. $\vec{X} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$ c_1, c_2 arbitrary constants

The fundamental matrix is

$$\Psi(t) = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix}$$

$$\Psi^{-1}(t) = \begin{pmatrix} -\sin t & \cos t + 2\sin t \\ \cos t & -2\cos t + \sin t \end{pmatrix}$$

$$\vec{X} = \Psi \vec{c} + \Psi \underbrace{\int_0^t \Psi^{-1} g(s)\, ds}$$

$$= \int_0^t \begin{pmatrix} -\sin s & \cos s + 2\sin s \\ \cos s & -2\cos s + \sin s \end{pmatrix} \begin{pmatrix} \csc s \\ \sec s \end{pmatrix} ds$$

$$= \int_0^t \begin{pmatrix} 2\tan s \\ \cot s - 2 + \tan s \end{pmatrix} ds$$

$$= \begin{pmatrix} -2 \ln \cos t \\ -2t + \ln \tan t \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$