Math 3270B Tutorial 6.11

Abel's Formula Let $x^1(x), x^2(x), ..., x^n(x)$ be solutions of the differential system $X' = A(x)X$ in I with $x_0 \in I$. Then for all $x \in I$,

$$W(x) = W(x_0)\exp\left(\int_{x_0}^{x} TrA(t)dt\right).$$

Example For the differential system,

$$u' = \begin{bmatrix} 0 & 1 \\ -\dfrac{2}{x^2+2x-1} & \dfrac{2(x+1)}{x^2+2x-1} \end{bmatrix} u, \quad x \neq -1 \pm \sqrt{2}.$$

First we see that

$$u^1(x) = \begin{bmatrix} x+1 \\ 1 \end{bmatrix}, u^2(x) = \begin{bmatrix} x^2+1 \\ 2x \end{bmatrix}$$

are two linearly independent solutions, and

$$W(u^1, u^2)(x) = \begin{vmatrix} x+1 & x^2+1 \\ 1 & 2x \end{vmatrix} = x^2 + 2x - 1,$$

$$\exp\left(\int_{x_0}^{x} TrA(t)dt\right) = \exp\left(\int_{x_0}^{x} \frac{2(t+1)}{t^2+2t-1}dt\right) = \frac{x^2+2x-1}{x_0^2+2x_0-1}, \quad x_0 \neq -1 \pm \sqrt{2},$$

hence the Abel's formula is satisfied.

***Review matrices**

Determine linearly independence;
Computer eigenvalues and corresponding eigenvectors, both real and complex cases.
You may refer to your linear algebra books for details.

In June 15th tutorial, we took examples in textbook Page 397 (real and opposite sign eigenvalues),401 (real and same sign eigenvalues) and 408 (complex eigenvalues) to study how to sketch a phase portrait for a linear 2x2 system, also the asymptotic behavier as $t \to \infty$.