MATH 3270B

Tutorial June 1st

Method of undetermined coefficients

1. $y'' - 2y' - 3y = -3te^{-t}$

   Sol: First solve homogeneous equation $y'' - 2y' - 3y = 0 \implies y_1 = e^{3t},\ y_2 = e^{-t}$. Consider particular solution in the form $Y(t) = At^2e^{-t} + Bte^{-t}$, and

   $$Y'(t) = (2A - B)te^{-t} - At^2e^{-t} + Be^{-t},$$

   $$Y''(t) = (2A - 2B)e^{-t} + At^2e^{-t} - (4A - B)te^{-t}.$$

   Plug $Y(t),\ Y'(t),\ Y''(t)$ into the equation and get $A = \frac{3}{8},\ B = \frac{3}{16}$, hence the general solution is $y(t) = c_1e^{3t} = c_2e^{-t} + \frac{3}{8}t^2e^{-t} + \frac{3}{16}te^{-t}$, where $c_1,\ c_2 \in \mathbb{R}$ are two constants.

2. $y'' + 2y' = 3 + 4\sin 2t$

   Sol: First solve homogeneous equation $y'' + 2y' = 0$, $y_1 = c, y_2 = e^{-2t}$, where c is a constant. Consider a particular solution in the form $Y(t) = At + B\sin 2t + C\cos 2t$, and

   $$Y' = A + 2B\cos 2t - 2C\sin 2t,$$

   $$Y'' = -4B\sin 2t - 4C\cos 2t.$$

   Plug $Y(t),\ Y'(t),\ Y''(t)$ into the equation and get $A = \frac{2}{3},\ B = C = -\frac{1}{2}$, hence the general solution is $y(t) = c_1 + c_2e^{-2t} + \frac{3t}{2} - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t$, where $c_1,\ c_2 \in \mathbb{R}$ are two constants.

3. Show that $y_1(t) = t^{1/2},\ \ y_2(t) = t^{-1}$ form a fundamental set of solutions of $2t^2y'' + 3ty' - y = 0,\ \ t > 0$.

   Sol: First we verify that $y_1,\ y_2$ are indeed solutions of $2t^2y'' + 3ty' - y = 0,\ \ t > 0$, then calculate the Wronskian $W = -\frac{3}{2}t^{-\frac{3}{2}}$, which is not zero since $t > 0$. Hence $y_1,\ y_2$ form a fundamental set of solutions.

4. **Euler equations** $t^2y'' - ty' + 5y = 0$

   Sol: $\alpha = -1,\ \beta = 5$, take $x = \ln t$, then

   $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0,$$

   we have $y_1(x) = e^x\cos 2x,\ \ y_2(x) = e^x\sin 2x$, hence the general solution is $y(t) = c_1t\sin(2\ln t) + c_2t\cos(2\ln t)$, where $c_1,\ c_2 \in \mathbb{R}$ are two constants.

5. $2t^2y'' - 5ty' + 5y = 0$

Sol: First divide 2 on both sides, and we have $\alpha = -\frac{5}{2},\ \beta = \frac{5}{2}$,take $x = \ln t$, then

$$\frac{d^2y}{dx^2} - 2\frac{7}{2}\frac{dy}{dx} + \frac{5}{2}y = 0,$$

we have $y_1(x) = e^x, y_2(x) = e^{\frac{5}{2}x}$, hence the general solution is $y(t) = c_1t + c_2t^{\frac{5}{2}}$, where $c_1,\ c_2 \in \mathbb{R}$ are two constants.

6. **Exact equation** $x^2y'' + xy' - y = 0$

Sol: First we check that this equation is exact: $P(x) = x^2,\ P''(x) = 2,\ Q(x) = x,\ Q'(x) = 1,\ R(x) = -1$. Hence $P''(x) - Q'(x) + R(x) = 0$, exact.
Then for some constant $C \in \mathbb{R}$, $C = P(x)y' + (Q(x) - P'(x))y \longrightarrow x^2y' - xy = C$, this is a 1st order ODE, one may solve it by using integrating factor. And get

$$y = Ct^{-1} + c_2t$$

, where $c_2 \in \mathbb{R}$ is another constant.