Tutorial 6.4.

* Reduction of order
* method of variation of parameter.

① $t^2 y'' - t(t+2)y' + (t+2)y = 2t^3$, $t>0$,
given $y_1 = t$ is a solution to the homo. eqn.

Sol. Assume the general solution $y(t) = u(t)\, y_1(t) = tu$

$y' = u + tu'$

$y'' = 2u' + tu''$

Substitute y, y', y'' into ①,

$\Rightarrow \quad u'' - u' = 2.$

Take $v = u'$, then $v' - v = 2$, $v = -2 + c_1 e^t$

Integrate v, then $u = -2t + c_1 e^t + c_2$

Hence $y = u y_1 = -2t^2 + c_1 t e^t + c_2 t$, $c_1, c_2 \in \mathbb{R}$.

② $xy'' - (1+x)y' + y = x^2 e^{2x}$, $x > 0$

given $y_1(x) = e^x$ is a solution of the homo. eqn.

Sol: Assume $y = uy_1$ is the general solution,

$y' = u'e^x + ue^x$

$y'' = u''e^x + 2u'e^x + ue^x$

Substitute y, y', y'' into ②,

$\Rightarrow \quad u'' + u' - \frac{u'}{x} = xe^x$

take $v = u'$, $v' + (1 - \frac{1}{x})v = xe^x$

then $v = \frac{1}{2}xe^x + c_1 xe^{-x}$

Integrate v,

$\Rightarrow \quad u = \frac{1}{2}(xe^x - e^x) + c_1(-xe^{-x} - e^{-x}) + C_2$.

Hence $y = ue^x = \frac{1}{2}(xe^{2x} - e^{2x}) + c_1(-x-1) + c_2 e^x$,

$c_1, c_2 \in \mathbb{R}$.

③ $x^2y'' - 3xy' + 4y = \sqrt{x}$ on $(0,\infty)$,

given $y_1(x) = x^2$ as a solution to the homogeneous eqn.

Sol: Assume $y = uy_1$ is the general solution to ①.

$y' = 2xu + x^2u'$

$y'' = 2u + 4xu' + x^2u''$

Substitute y, y', y'' into ③, we get

$$u'' + \frac{1}{x}u' = x^{-\frac{7}{2}},$$

take $v = u'$, then $v' + \frac{1}{x}v = x^{-\frac{7}{2}}$,

$$v = -\frac{2}{3}x^{-\frac{5}{2}} + \frac{c_1}{x}$$

Integrate $v \Rightarrow u = \frac{4}{9}x^{-\frac{3}{2}} + c_1\ln(x) + c_2$

Hence $y(x) = x^2u = \frac{4}{9}x^{\frac{1}{2}} + c_1x^2\ln(x) + c_2x^2$,

$c_1, c_2 \in \mathbb{R}$.

④ $y'' - 2y' - 3y = xe^{-x}$

Sol: It's easy to see that $y_1 = e^{-x}$, $y_2 = e^{3x}$,

To find a particular solution $Y(x)$,

$Y(x) = y_1 u_1 + y_2 u_2$, $r(x) = xe^{-x}$, $W(y_1, y_2) = 4e^{2x}$

$$u_1 = -\int \frac{y_2 r(x)}{W(y_1, y_2)} dx$$

$$= -\frac{x^2}{8}$$

$$u_2 = \int \frac{y_1 r(x)}{W(y_1, y_2)} dx = -\frac{x}{16} e^{-4x} - \frac{1}{64} e^{-4x}$$

$$\therefore Y(x) = -\frac{x^2}{8} e^{-x} - \frac{x}{16} e^{-x} - \frac{1}{64} e^{-x}$$

Hence general solution $y(x) = c_1 e^{-x} + c_2 e^{3x} - \frac{x^2}{8} e^{-x} - \frac{x}{16} e^{-x} - \frac{1}{64} e^{-x}$.