Math 3270 B.

Tutorial 5.18

* Method of integrating factor ① ②
* Seperable equations ③ ④
* Homogeneous equation. ⑤

① $\begin{cases} ty' + 2y = 2\sin t & t>0 \\ y(\frac{\pi}{2}) = 1 \end{cases}$

Sol. $y' + \frac{2}{t}y = \frac{2}{t}\sin t$

take $\mu(t) = e^{\int \frac{2}{t}dt} = t^2$

$\Rightarrow \frac{d(t^2 y)}{dt} = 2t\sin t$ , integrate on both sides,

$t^2 y = 2\sin t - 2t\cos t + C$

$y = \frac{2}{t^2}\sin t - \frac{2}{t}\cos t + \frac{C}{t^2}$

$\because y(\frac{\pi}{2}) = 1$

$\Rightarrow C = \frac{\pi^2}{4} - 2$

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② $\begin{cases} ty' + (t+1)y = t & t>0 \\ y(\ln 2) = 1 \end{cases}$

Sol.

$y' + (1+\frac{1}{t})y = 1$, take $\mu(t) = e^{\int 1+\frac{1}{t}\,dt} = te^t$

$\Rightarrow \frac{d(te^t y)}{dt} = te^t$, integrate on both sides :

$$te^t y = te^t - e^t + C$$

$$\Rightarrow y = 1 - \frac{1}{t} + \frac{C}{te^t}$$

Since $y(\ln 2) = 1$, $\Rightarrow C = 2$

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③ $\begin{cases} y' = \dfrac{x}{y+x^2 y} \\ y(0) = -2 \end{cases}$

Sol. Rewrite the equation : $y\,dy = \frac{x}{1+x^2}\,dx$

integrate on both sides : $\frac{1}{2}y^2 = \frac{1}{2}\ln(1+x^2) + C$

$$\Rightarrow y = \pm\sqrt{\ln(1+x^2)+C}$$

$\because$ $y(0) = -2$, we take "+" sign,

and $C = 4$.

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④ $\begin{cases} \sin 2x\, dx + \cos 3y\, dy = 0 \\ y(\frac{\pi}{4}) = \frac{\pi}{9} \end{cases}$

we've made change to this initial condition
So that "arcsin" function make sense.

Sol.

$$\cos 3y\, dy = -\sin 2x\, dx$$

integrate on both sides: $\frac{1}{3}\sin 3y = \frac{1}{2}\cos 2x + C$

$$\Rightarrow y = \frac{1}{3}\arcsin\left(\frac{3}{2}\cos 2x + C\right)$$

$$y(\frac{\pi}{4}) = \frac{\pi}{9} \Rightarrow C = \frac{\sqrt{3}}{2}$$ #

⑤ $$\frac{dy}{dx} = \frac{y-4x}{x-y}$$

Sol: Rewrite the equation as $\frac{dy}{dx} = \frac{\frac{y}{x}-4}{1-\frac{y}{x}}$

take $v(x) = \frac{y}{x}$, then $y = v(x)x$,

$y' = v(x) + xv'(x)$

Hence $v + xv' = \frac{v-4}{1-v}$.

Do seperation: $\frac{1-v}{(v+2)(v-2)} = \frac{1}{x}dx$,

and $\frac{1-v}{(v+2)(v-2)} = -\frac{3}{4}\cdot\frac{1}{v+2} - \frac{1}{4}\cdot\frac{1}{v-2}$.

Hence we have :

$$\left(-\frac{3}{4}\cdot\frac{1}{v+2}-\frac{1}{4}\cdot\frac{1}{v-2}\right)dv=\frac{1}{x}dx.$$

Integrate on both sides : $-\frac{3}{4}\ln|v+2|-\frac{1}{4}\ln|v-2|=\ln|x|+C$

$$\Rightarrow \ln|v+2|^3|v-2||x|^4=C$$

$\because$ $v=\frac{y}{x}$ , change back to $y$, we have

$$\Rightarrow \ln|y+2x|^3|y-2x|=C, \quad C: \text{constant}$$

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