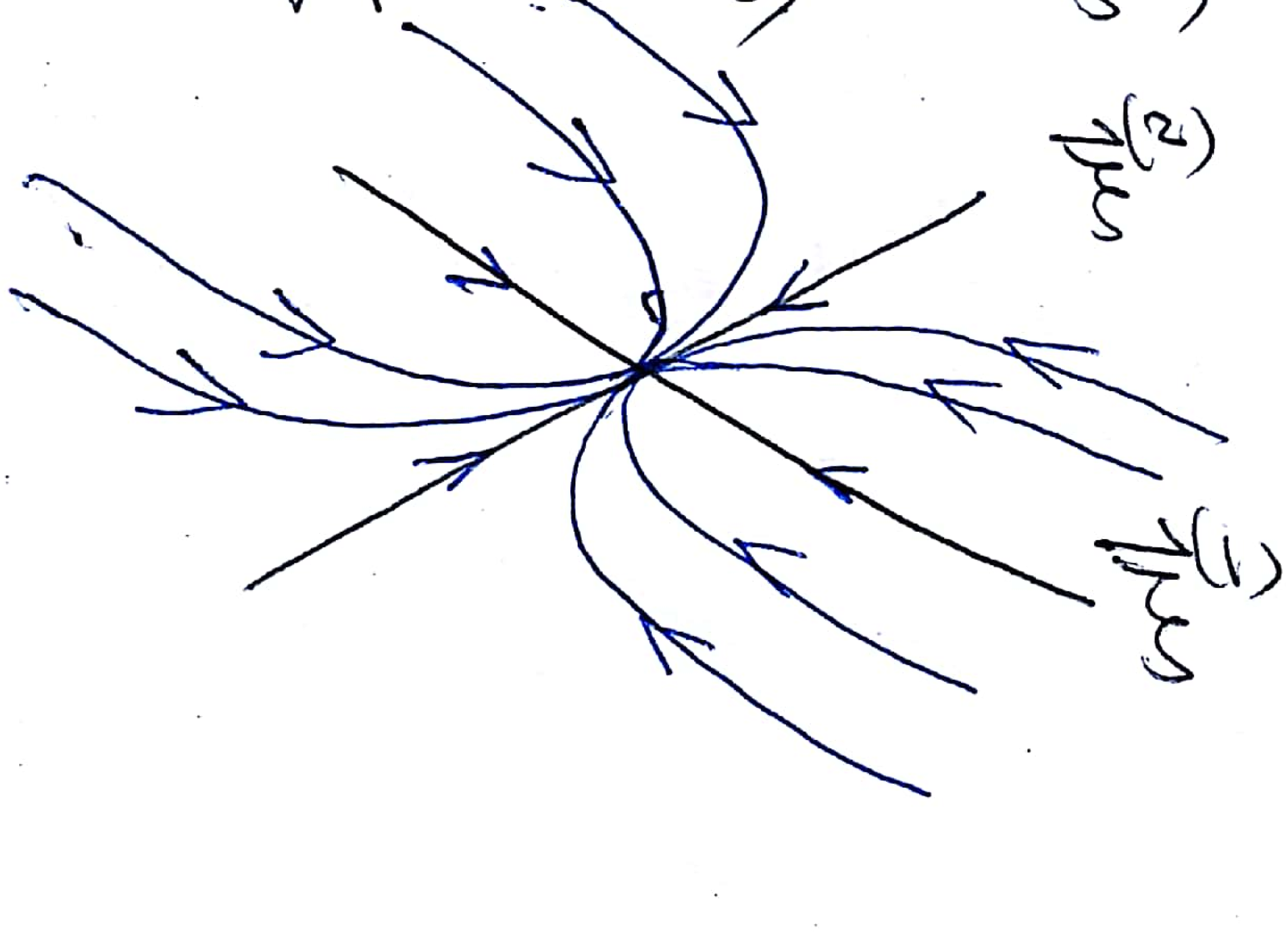
$$\frac{d}{dt}\vec{x} = A_{2\times2}\vec{x}$$

λ_1, λ_2 eigenvalues

phase portrait

<u>Case I</u> real, unequal, of same sign

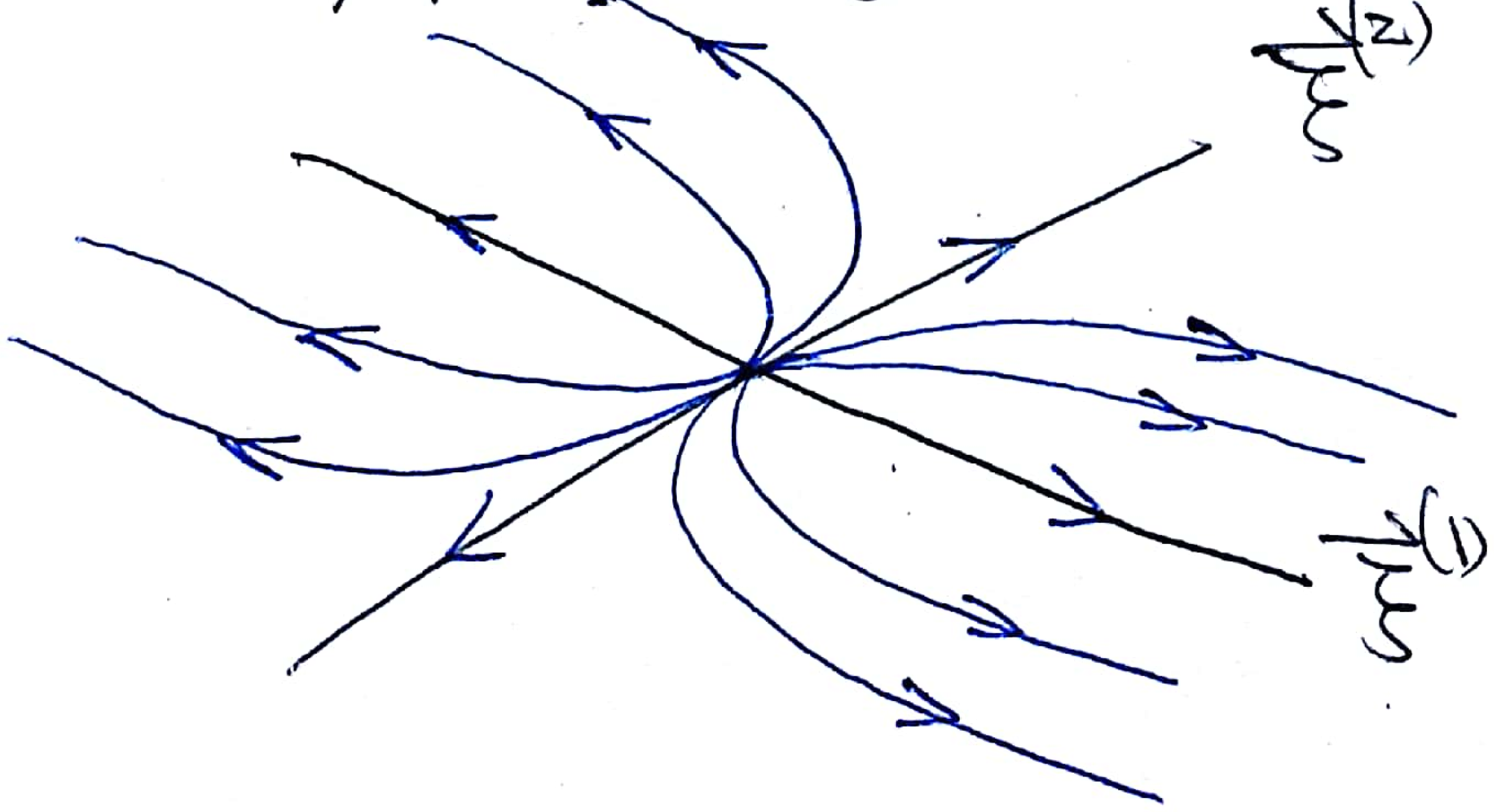
- $\lambda_1 < \lambda_2 < 0$ $\quad \vec{\xi}^{(1)}, \vec{\xi}^{(2)}$ eigenvectors



Node (Nodal sink)

asymptotically stable

- $\lambda_1 > \lambda_2 > 0$

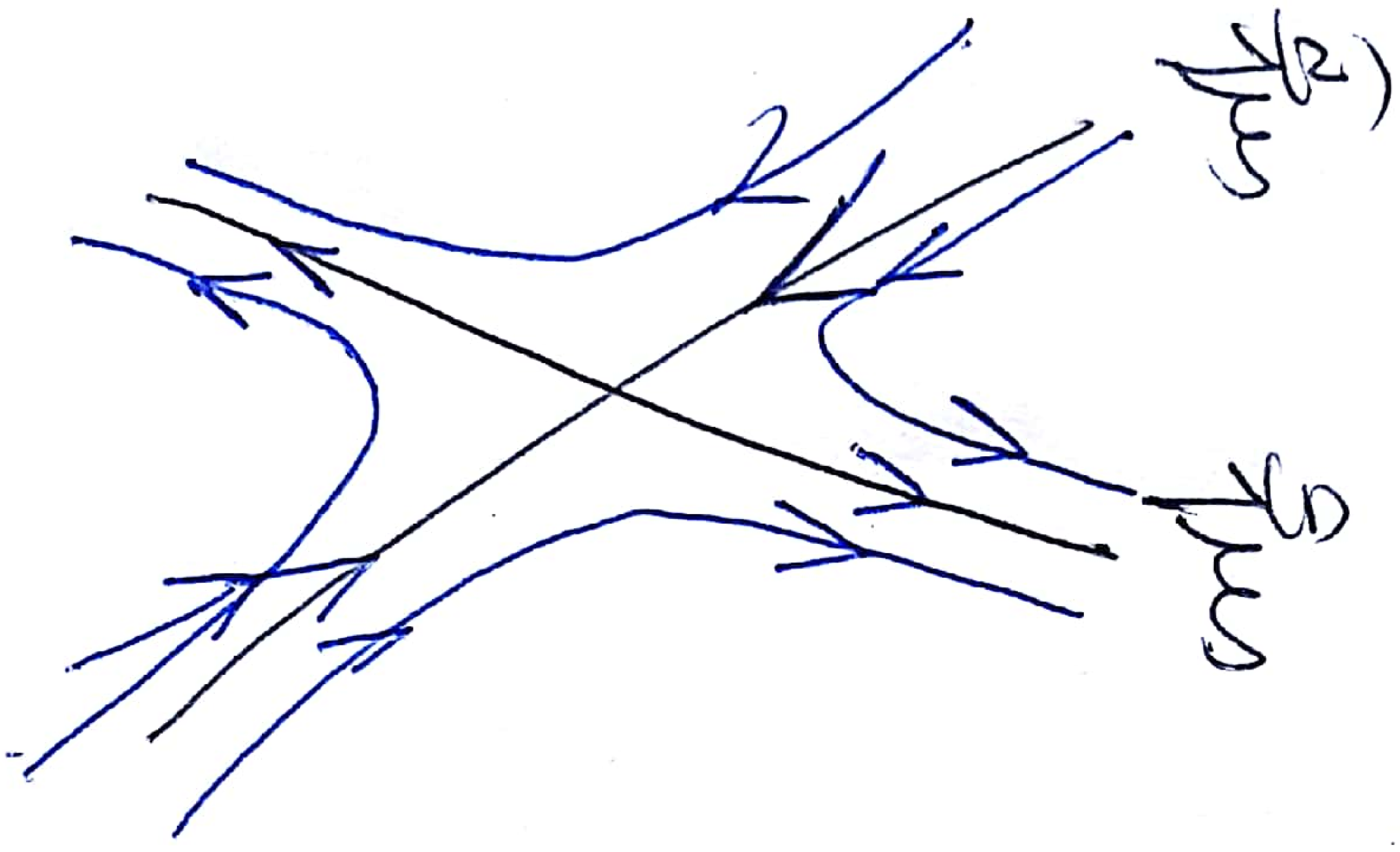


Node (Nodal source)

Unstable

Case II real, opposite sign

$\lambda_2 < 0 < \lambda_1$ $\quad \vec{\xi}^{(1)} \quad \vec{\xi}^{(2)}$ eigenvectors

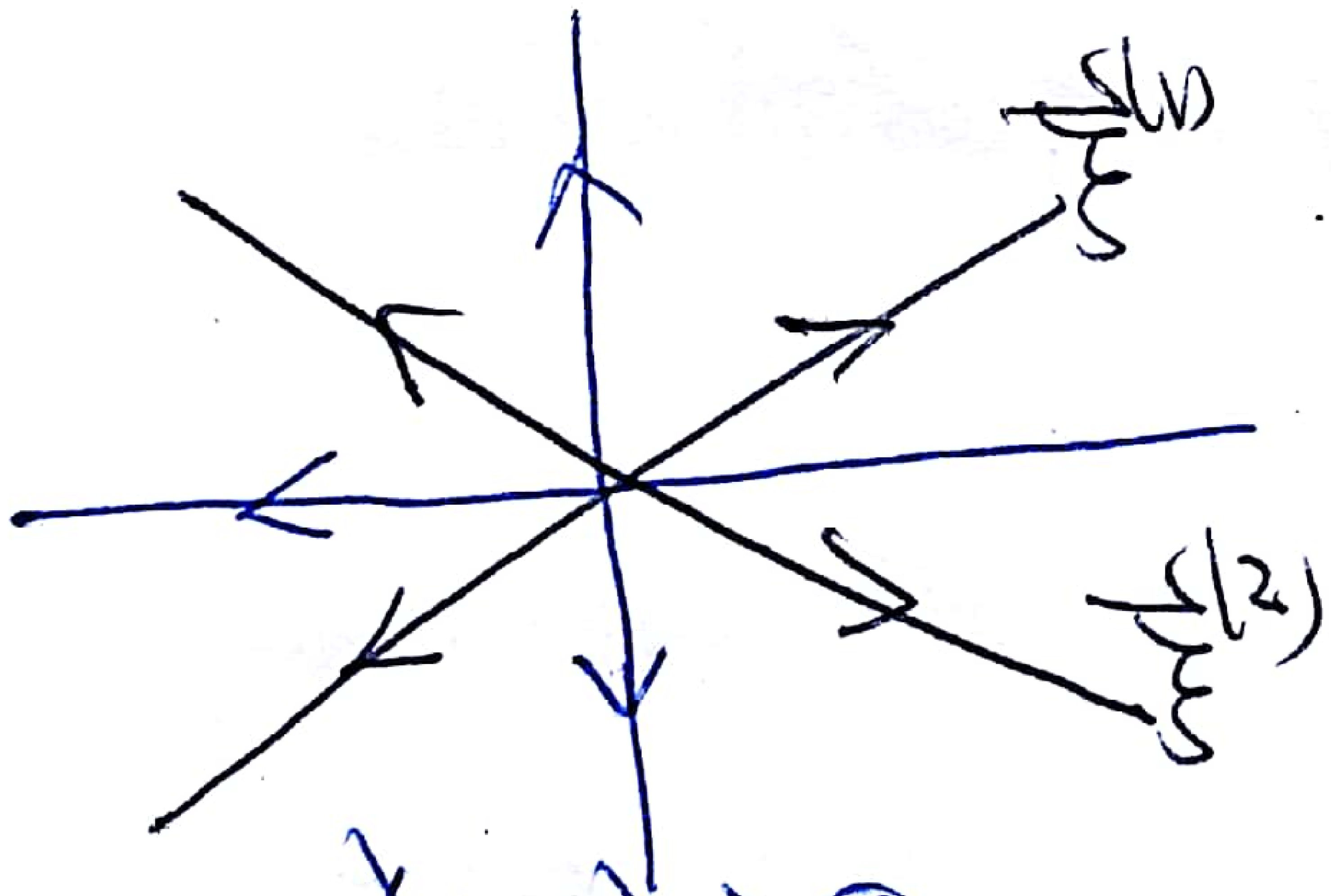


Saddle point

Unstable

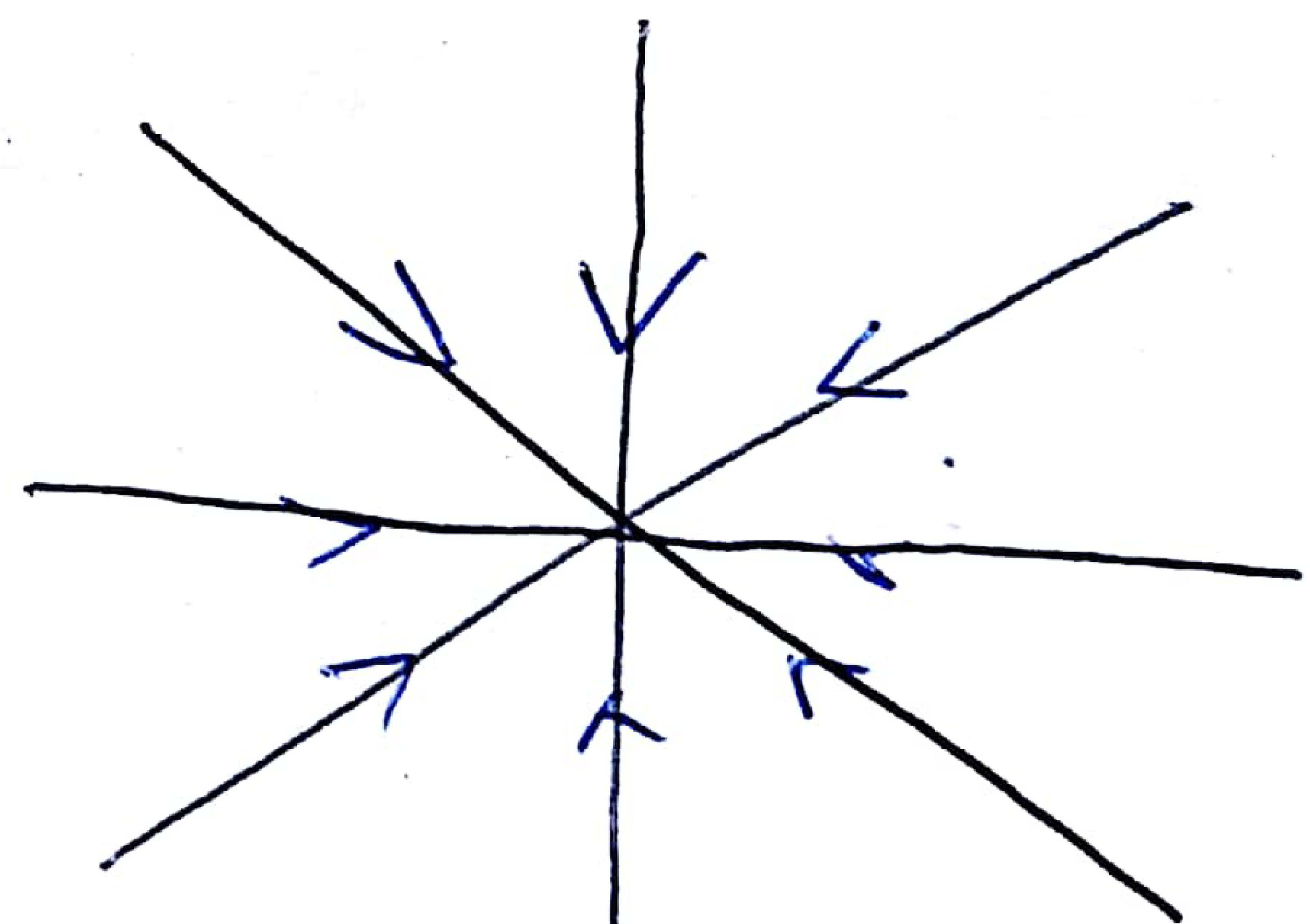
Case III real, equal $(\lambda_1 = \lambda_2)$

(a) $\vec{\xi}^{(1)}, \vec{\xi}^{(2)}$ linearly independent eigenvectors ~~vectors~~



$\lambda_1 = \lambda_2 > 0$

Proper Node
(Star point)
asymptotically stable

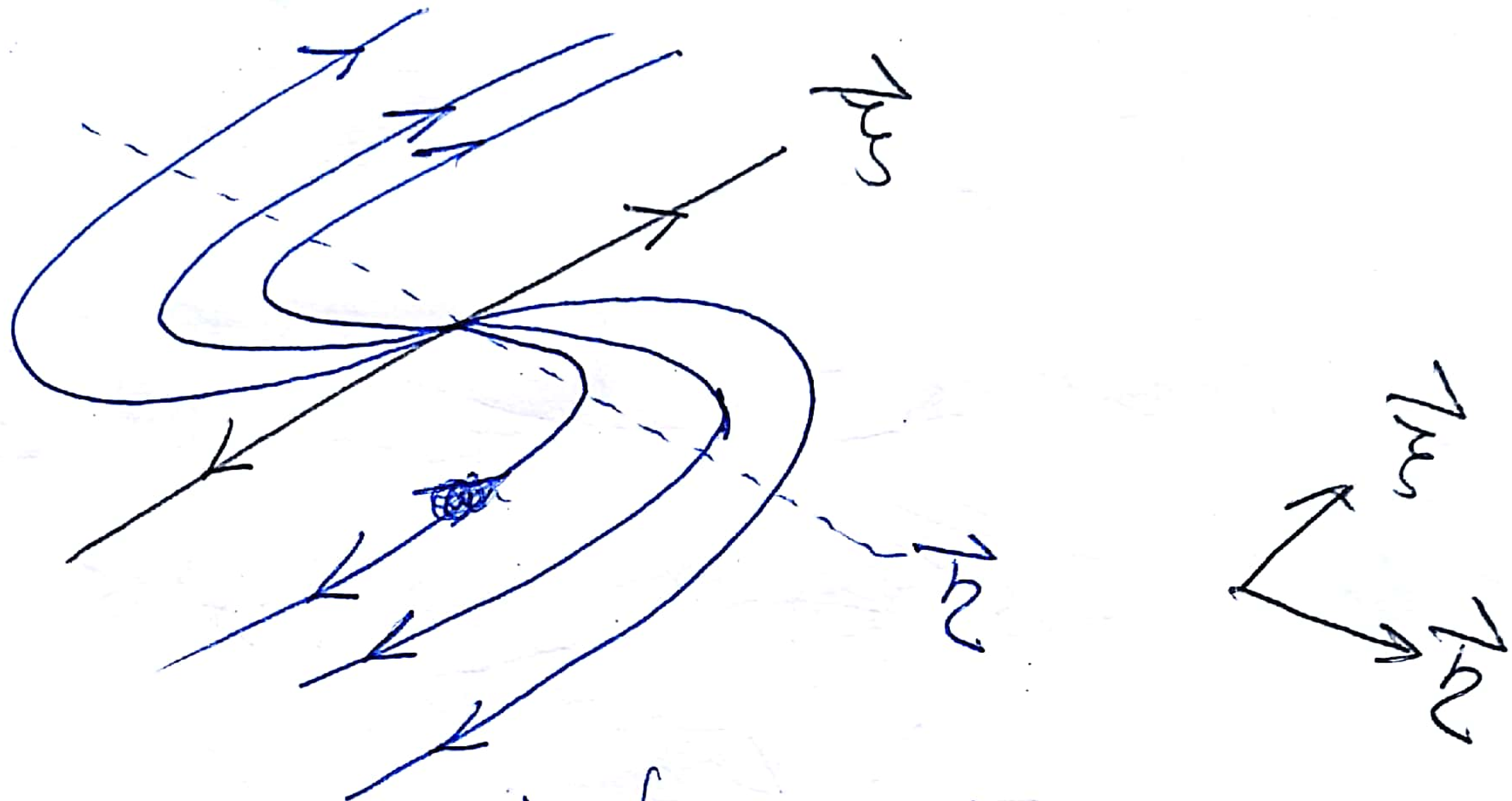


$\lambda_1 = \lambda_2 < 0$

Proper Node
(Star points)
Unstable

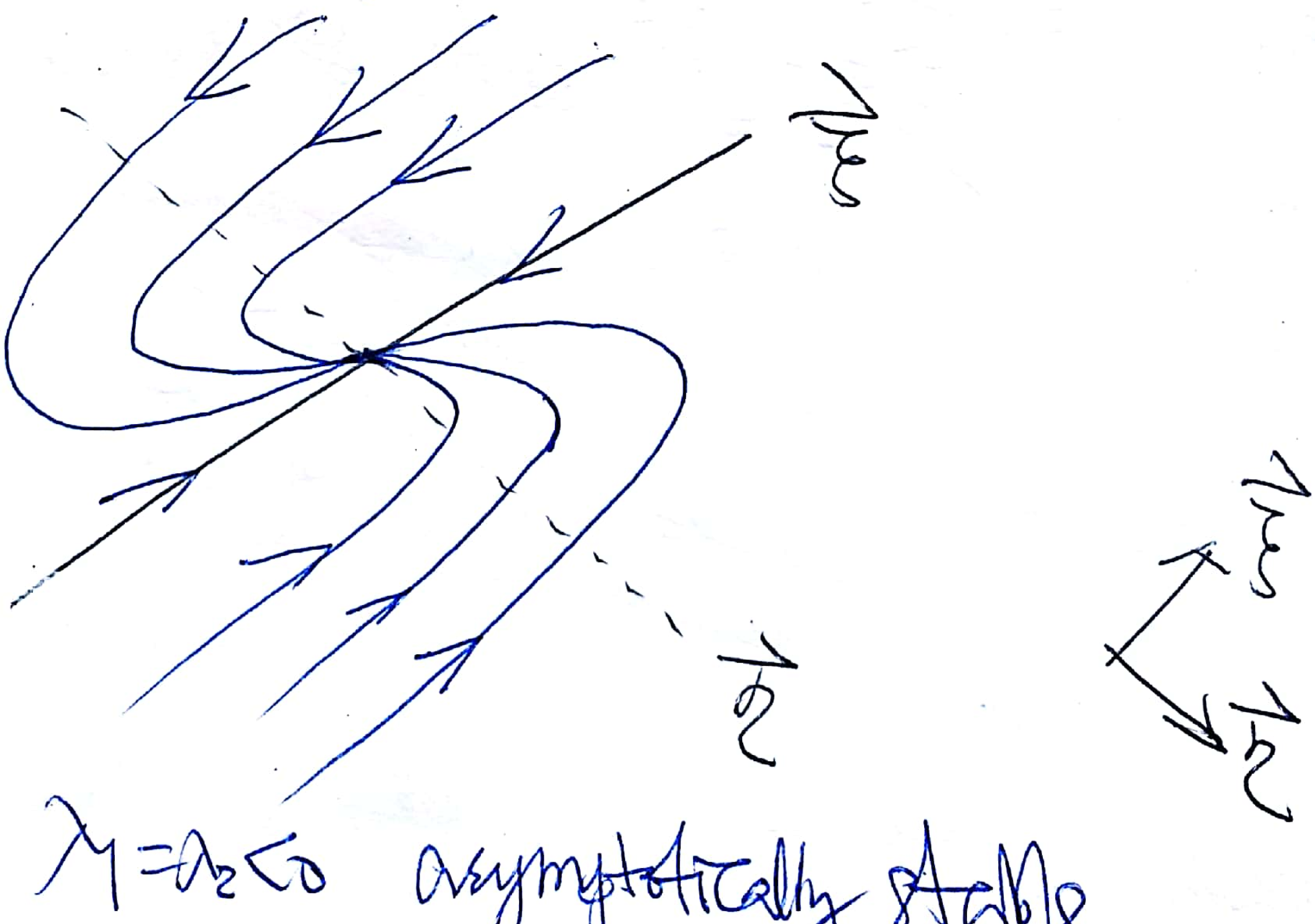
(b) $\vec{\xi}$ only one linearly independent eigenvector

$(A-\lambda I)\vec{\eta}=\vec{\xi}$



Improper node (degenerate node)

$\lambda_1=\lambda_2>0$, Unstable



$\lambda_1=\lambda_2<0$ asymptotically stable

Case IV Complex eigenvalues with Re $\neq 0$.

$\lambda_1 = \alpha + i\beta \qquad \lambda_2 = \alpha - i\beta \qquad \alpha \neq 0.$

$\vec{\xi}^{(1)} = \vec{u} + i\vec{v} \qquad \vec{\xi}^{(2)} = \vec{u} - i\vec{v}$

$$\vec{\xi}^{(1)} e^{\lambda_1 t} = (\vec{u} + i\vec{v}) e^{\alpha t} (\cos\beta t + i\sin\beta t)$$

$$= (\vec{u}\cos\beta t - \vec{v}\sin\beta t) e^{\alpha t}$$

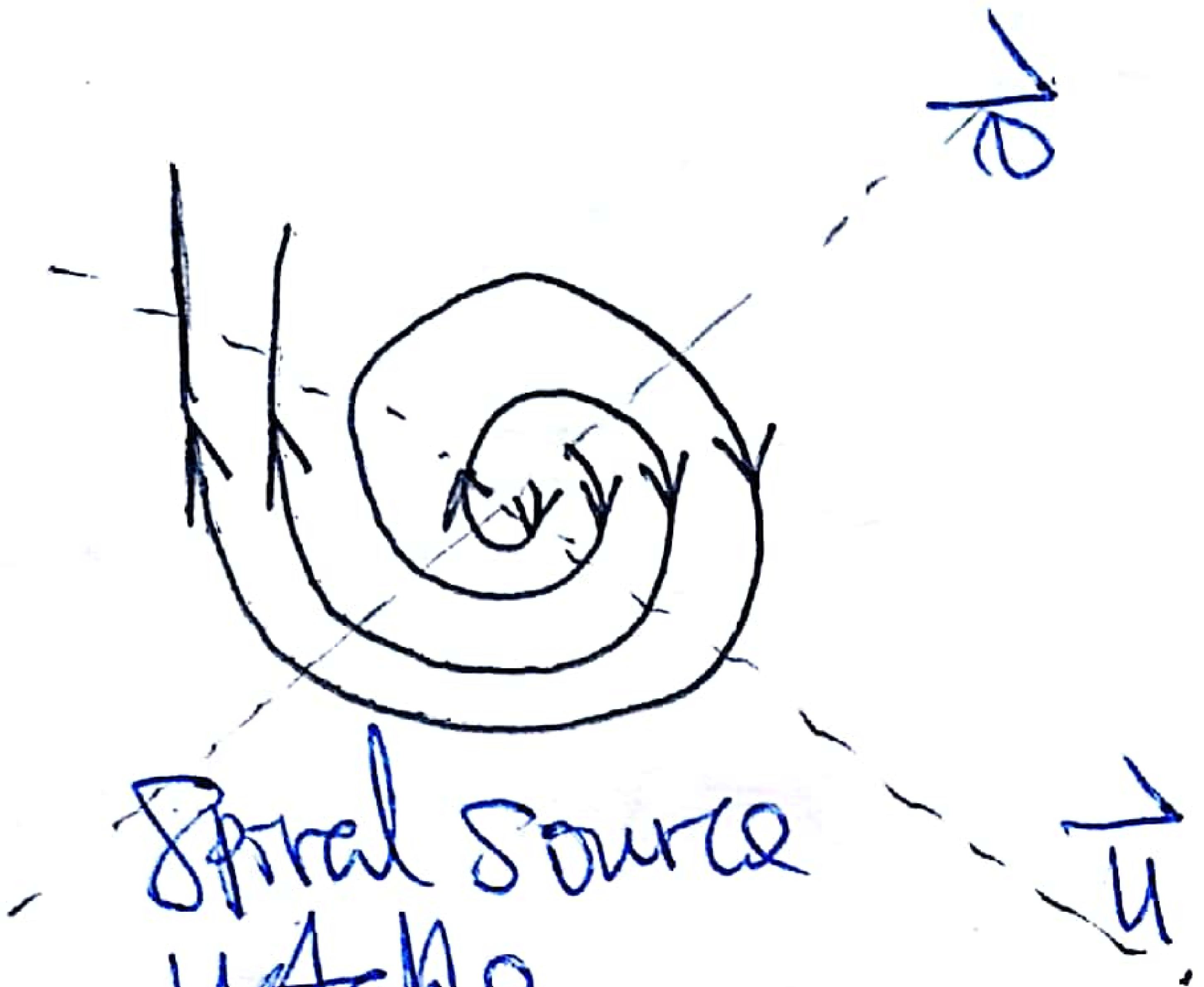
$$+ i(\vec{u}\sin\beta t + \vec{v}\cos\beta t) e^{\alpha t}.$$

$$\vec{x}(t) = C_1 (\vec{u}\cos\beta t - \vec{v}\sin\beta t) e^{\alpha t}$$

$$+ C_2 (\vec{u}\sin\beta t + \vec{v}\cos\beta t) e^{\alpha t}$$

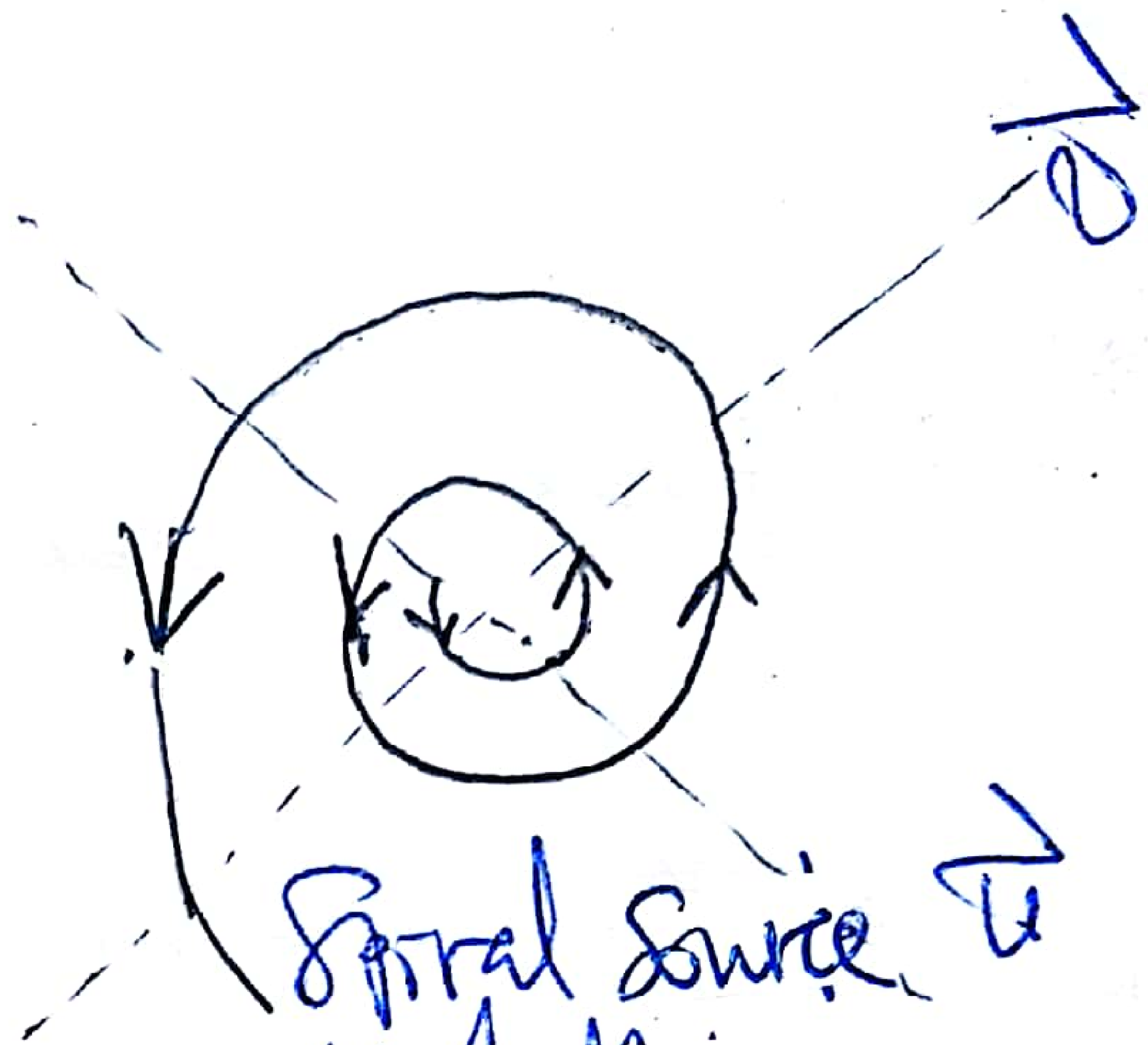
- $\alpha > 0$ expand
  $\alpha < 0$ shrink
- $\beta > 0$ rotate $\vec{v} \to \vec{u}$
  $\beta < 0$ rotate $\vec{u} \to \vec{v}$

Spiral point



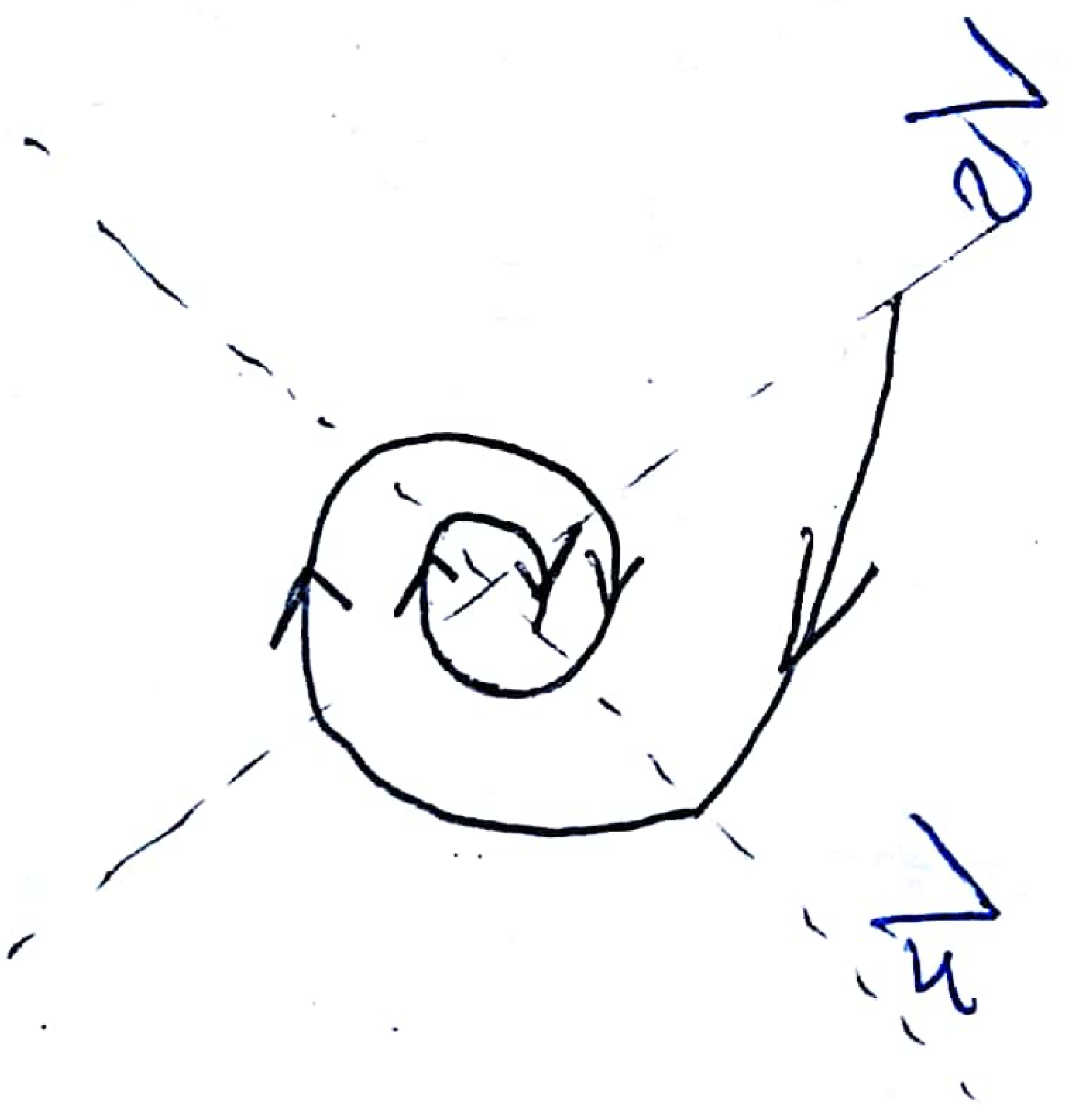
Spiral source
Ustable

$\alpha>0,\ \beta>0,\ \vec{v}\to\vec{u}$



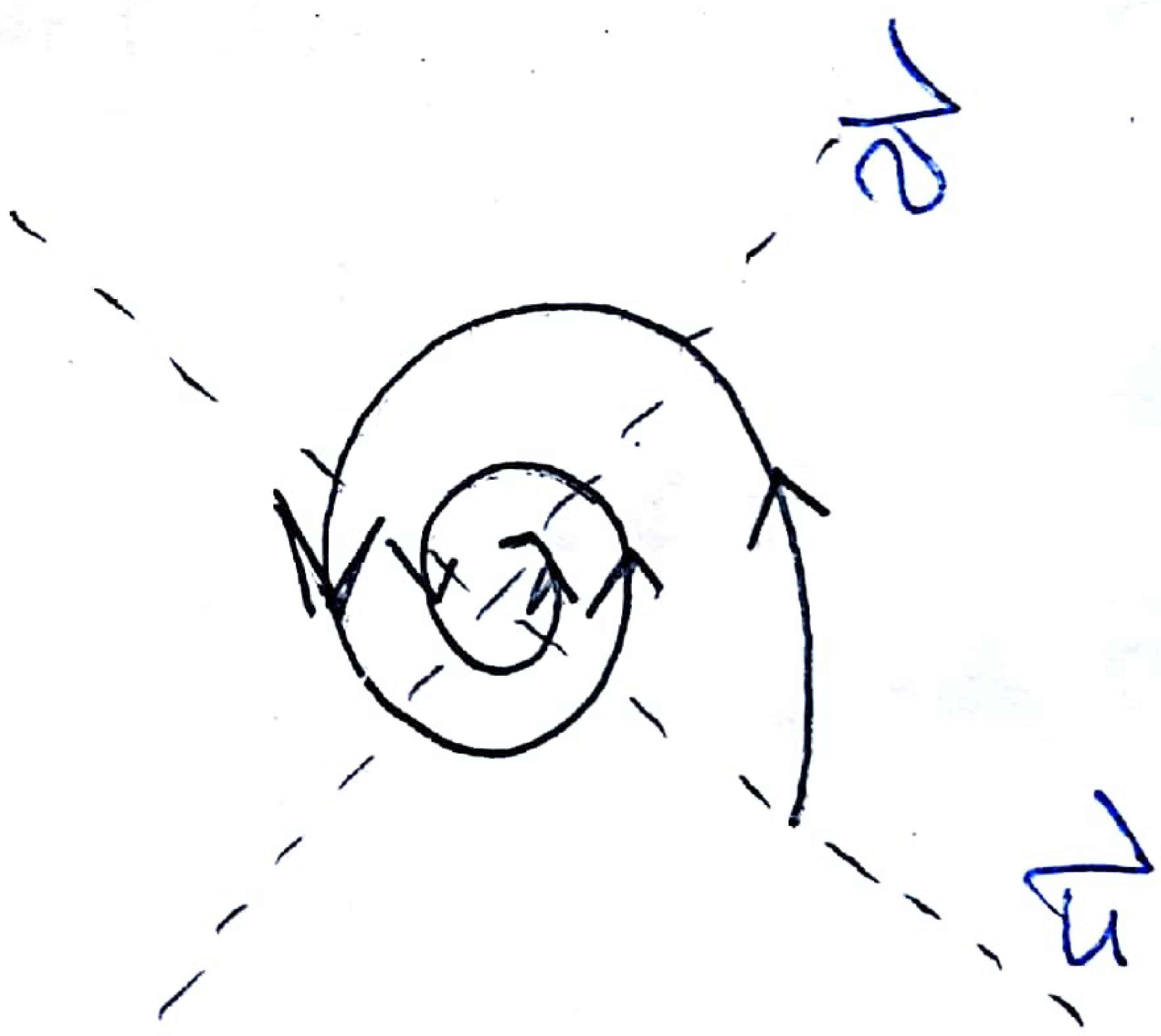
Spiral source
Ustable

$\alpha>0,\ \beta<0,\ \vec{u}\to\vec{v}$



$\alpha<0\quad \beta>0\quad \vec{v}\to\vec{u}$

Spiral Sink
asymptotically stable



$\alpha<0,\ \beta<0,\ \vec{u}\to\vec{v}$

Spiral Sink
asymptotically stable

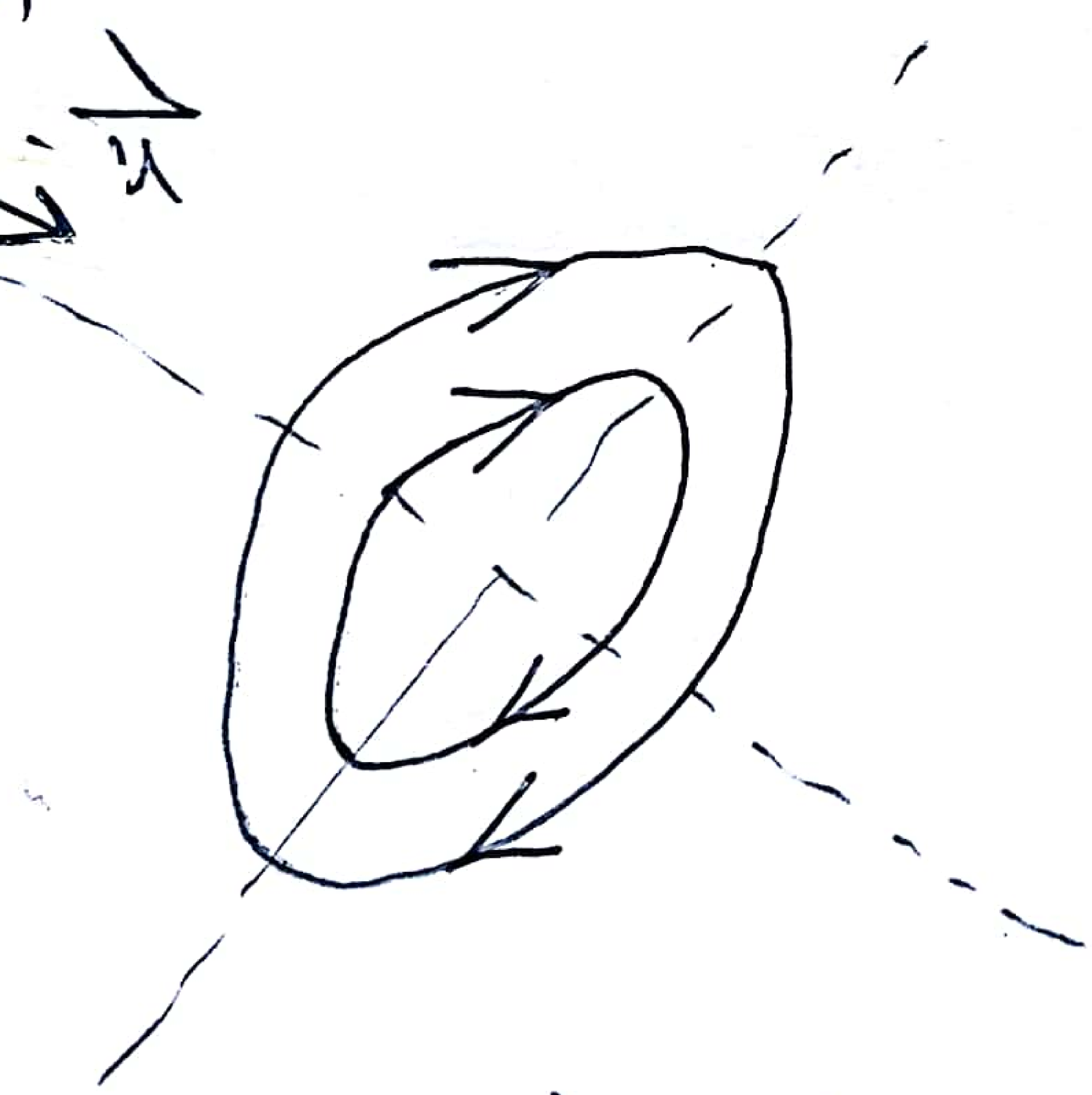
Case V Pure Imaginary eigenvalues

$\lambda_1 = i\beta \qquad \lambda_2 = -i\beta$

$\vec{\xi}^{(1)} = \vec{u} + i\vec{v}, \qquad \vec{\xi}^{(2)} = \vec{u} - i\vec{v}$

$$\vec{x}(t) = c_1\left(\vec{u}\cos\beta t - \vec{v}\sin\beta t\right) + c_2\left(\vec{u}\sin\beta t + \vec{v}\cos\beta t\right)$$

- $\beta > 0$, rotate $\vec{v} \to \vec{u}$
- $\beta < 0$, rotate $\vec{u} \to \vec{v}$



$\beta > 0 \quad \vec{v} \to \vec{u}$

$\beta > 0 \quad \vec{u} \to \vec{v}$

Center Stable
but Not asymptotically Stable

Conclusion Stability of $\frac{d}{dt}\vec{x} = A_{2\times2}\vec{x}$

$\det A \neq 0$.

Eigenvalues	Type of Critical point	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically Stable
$\lambda_1 < 0 < \lambda_2$	Saddle Point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or Improper Node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or Improper Node	Asymptotically Stable
$\lambda_1, \lambda_2 = \alpha + i\beta$		
$\alpha > 0$	Spiral point	Unstable
$\alpha < 0$	Spiral point	Asymptotically Stable
$\lambda_1, \lambda_2 = i\beta$	Center	Stable

$\mathrm{Re}\,\lambda_1$ & $\mathrm{Re}\,\lambda_2 < 0$

$\iff$ asymptotically stable (A)

$\mathrm{Re}\,\lambda_1$ or $\mathrm{Re}\,\lambda_2 > 0$ (B)

$\iff$ Unstable

Pure Imaginary

$\implies$ Stable, but Not asymptotically stable

(A) & (B) also hold

for locally linear system !!

9.3 Locally linear system

$$x' = f(\vec{x}) \qquad \text{(NS)}$$

$\vec{x}_0$ critical point of $f(x)$, i.e. $f(x_0) = 0$.

Locally linear near $\vec{x}_0$:

- $f(\vec{x}) = A(x - x_0) + g(x)$,
- $\det A \neq 0$
- $\lim_{x \to x_0} \frac{\|g(x)\|}{\|x - x_0\|} = 0$

Linearized system of (NS) near x_0:

$$\vec{x}' = A(\vec{x} - \vec{x}_0). \qquad \text{(LS)}$$

Note:

$$A = \nabla f(x_0). \qquad \vec{x}' = \nabla f(x_0)(x - x_0).$$

Ex 4 Find the critical points and the corresponding linearized system near the critical points

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\omega^2 \sin x - \gamma y \end{cases}$$

Sol. $\begin{cases} y = 0 \\ \omega^2 \sin x + \gamma y = 0 \end{cases} \Longrightarrow \begin{cases} x = k\pi \\ y = 0. \end{cases} \quad k \in \mathbb{Z}.$

Therefore, the critical points are

$(k\pi, 0)$, $k = 0, \pm 1, \pm 2, \ldots$

Note

$$f(x,y) = \begin{pmatrix} y \\ -\omega^2 \sin x - \gamma y \end{pmatrix}$$

$$\nabla f(x,y) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos x & -\gamma \end{pmatrix}$$

$$\nabla f(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\omega^2 \cos k\pi & -\gamma \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ (-1)^{k+1}\omega^2 & -\gamma \end{pmatrix}.$$

The linearized system near $(k\pi, 0)$ is

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \nabla f(k\pi, 0) \begin{pmatrix} x - k\pi \\ y \end{pmatrix}.$$

$$= \begin{pmatrix} 0 & 1 \\ (-1)^{k+1}\omega^2 & -\gamma \end{pmatrix} \begin{pmatrix} x - k\pi \\ y \end{pmatrix}, \quad k \in \mathbb{Z}.$$

Thm 9.3.1 $\quad x' = f(x) \quad$ (NS)

$x' = A(x - x_0), \quad A = \nabla f(x_0), \quad \det A \neq 0$

λ_1, λ_2 eigenvalues of A

	linearized system		Locally Linear Systems	
	Type	Stability	Type	Stability
$\lambda_1 > \lambda_2 > 0$	N	Unstable	N	Unstable
$\lambda_1 < \lambda_2 < 0$	N	Asymptotically	N	Asymptotically
$\lambda_1 < 0 < \lambda_2$	SP	Unstable	SP	Unstable
$\lambda_1 = \lambda_2 > 0$	PN or IN	Unstable	N or SpP	Unstable
$\lambda_1 = \lambda_2 < 0$	PN or IN	Asymptotically	N or SpP	Asymptotically
$\lambda_1, \lambda_2 = \alpha + i\beta$				
$\alpha > 0$	SpP	Unstable	SpP	Unstable
$\alpha < 0$	SpP	Asymptotically	SpP	Asymptotically

$\lambda_1 \lambda_2 = i\beta$	C	Stable	C or SpP	Indetermined

N = Node, SP = Saddle Point

PN = Proper node IN = Improper node

S_pP = Spiral point C = Center.

Conclusion

- λ_1, λ_2 not pure Imaginary

$\Longrightarrow$ Stability of locally linear system

= Stability of the linearized system

- $\lambda_1 \neq \lambda_2$ not pure Imaginary

$\Longrightarrow$ Stability & Type of locally linear

= Stability & Type of linearized

Go back to Ex 1.

$$\begin{cases} \dfrac{dx}{dt} = y \\[2ex] \dfrac{dy}{dt} = -w^2 \sin x - \gamma y \end{cases} \qquad (\gamma > 0,\ w > 0)$$

<u>Critical points</u>

$(k\pi, 0)$, $k \in \mathbb{Z}$

<u>linearized system</u> near $(k\pi, 0)$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ (-1)^{k+1} w^2 & -\gamma \end{pmatrix} \begin{pmatrix} x - k\pi \\ y \end{pmatrix}.$$

- $k = 0$.

$$A = \begin{pmatrix} 0 & 1 \\ -w^2 & -\gamma \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ -w^2 & -\gamma-\lambda \end{vmatrix} = \lambda^2 + \gamma\lambda + w^2 = 0$$

$$\lambda = \frac{-\gamma \pm \sqrt{\gamma^2 - 4w^2}}{2}.$$

(i) $\gamma \geq 2w$, then $\lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4w^2}}{2} < 0$,

(ii) $\gamma < 2w$. $\lambda = -\dfrac{\gamma}{2} \pm i\sqrt{4w^2 - \gamma^2}$

In Conclusion

$Re\,\lambda_1$ & $Re\,\lambda_2 < 0$.

$\Rightarrow$ (0,0) is an asymptotically stable critical point.

② $k=1$

$$A=\begin{pmatrix} 0 & 1 \\ w^2 & -\gamma \end{pmatrix}$$

$$\begin{vmatrix} -\lambda & 1 \\ w^2 & -\gamma-\lambda \end{vmatrix} = \lambda^2+\gamma\lambda-w^2=0,$$

$$\lambda_1=\frac{-\gamma+\sqrt{\gamma^2+4w^2}}{2}, \quad \lambda_2=\frac{-\gamma-\sqrt{\gamma^2+4w^2}}{2}$$

$\lambda_2<0<\lambda_1$

$\Rightarrow$

(0,0) is a saddle point. unstable

Ex 2

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\sin x \end{cases}$$

Critical points

$$(k\pi, 0), \quad k = 0, \pm 1, \pm 2, \dots$$

linearized system

$$f(x,y) = \begin{pmatrix} y \\ -\sin x \end{pmatrix}, \quad \nabla f(k\pi, 0) = \begin{pmatrix} 0 & 1 \\ -\cos k\pi & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ (-1)^{k+1} & 0 \end{pmatrix}.$$

$\rightarrow$

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ (-1)^{k+1} & 0 \end{pmatrix} \begin{pmatrix} x - k\pi \\ y \end{pmatrix}.$$

- $k = 1$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = -1$$

$\Longrightarrow$ $(\pi, 0)$ is a saddle point

• $k=0$

$$A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2+1=0$$

$\lambda_1 = i, \quad \lambda_2 = -i$

$(0,0)$ is centor for linearized system

Not clear for the nonlinear system!

Study of the linearized system is not sufficient for the stability of the original nonlinear system!!

9.6 Liapnov's Second Method

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -\sin x \end{cases}$$

Critical point $(0,0)$.

$$\sin x \frac{dx}{dt} + y \frac{dy}{dt} = y \sin x - y \sin x = 0$$

$$\parallel$$

$$\frac{d}{dt}\left(-\cos x + \frac{y^2}{2}\right) = \frac{d}{dt}\left(1 - \cos x + \frac{y^2}{2}\right)$$

$\Longrightarrow$

$$1 - \cos x(t) + \frac{y^2(t)}{2} = 1 - \cos x(0) + \frac{y^2(0)}{2}.$$

We want study

(x, y) near $(0, 0)$

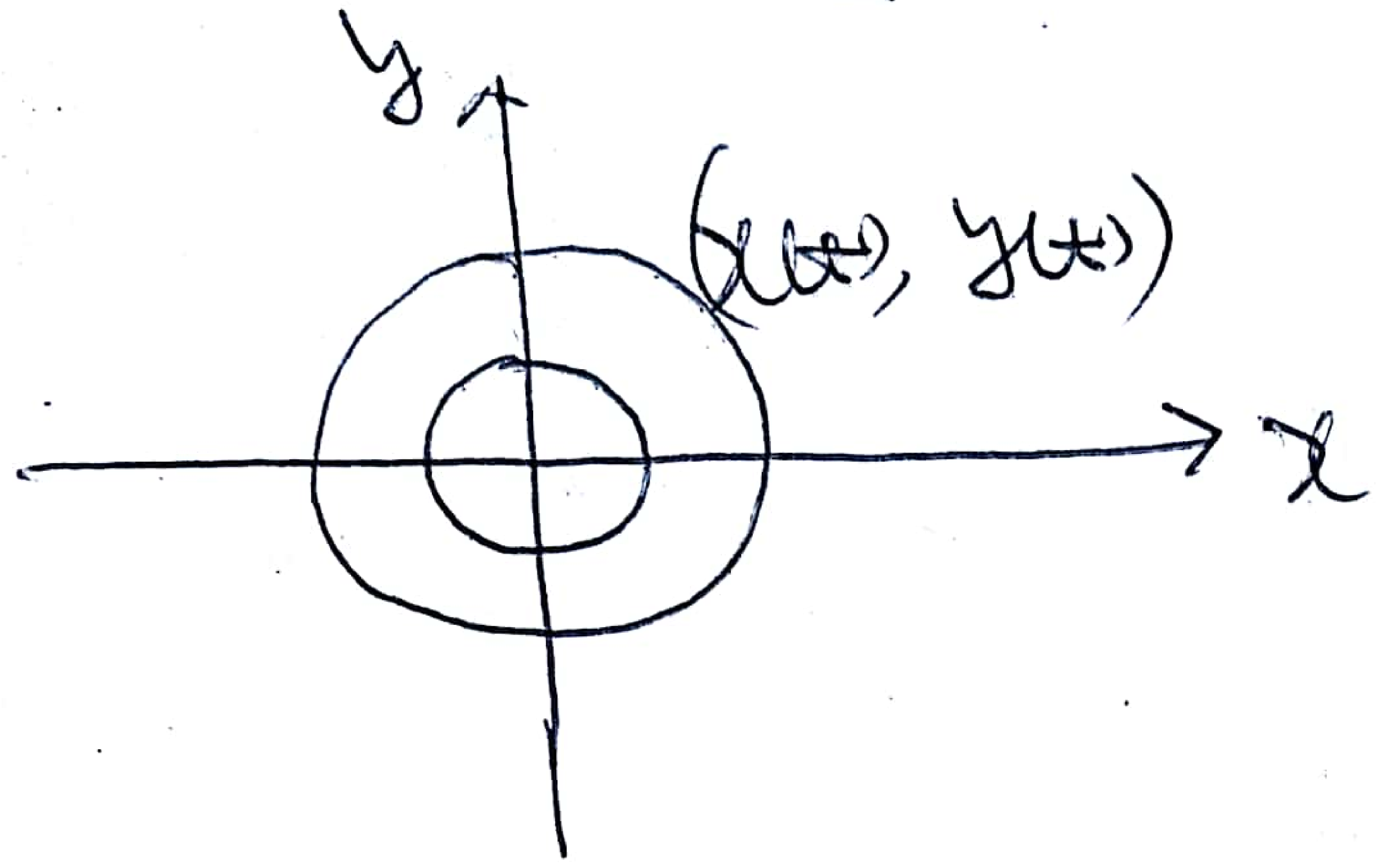
$$1 - \cos x \approx 1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$

$$\approx \frac{x^2}{2}$$

$\Longrightarrow$

Near $(0,0)$

$$\frac{x^2(t)}{2} + \frac{y^2(t)}{2} \approx C_0$$



$\Rightarrow$ $(0,0)$ is stable!

$$V(x,y) \triangleq 1 - \cos x + \frac{y^2}{2}$$

$$f(t) \triangleq V(x(t), y(t))$$

$$f'(t) = \left(1 - \cos x(t) + \frac{y^2(t)}{2}\right)'$$

$$= \sin x(t)\, x'(t) + y(t)\, y'(t)$$

$$= \sin x(t)\, y(t) + y(t)\,(-\sin x(t))$$

$$= 0.$$

General System

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases} \qquad (GS)$$

D: an open domain,

$(0,0) \in D$

$V: D \to \mathbb{R}$

Positive definite: $V(0,0)=0$, $V(x,y)>0$, $(x,y)\in D \setminus (0,0)$,

Negative definite: $V(0,0)=0$, $V(x,y)<0$, $(x,y)\in D \setminus (0,0)$,

Positive Semidefinite: $V(0,0)=0$, $V(x,y)\geq 0$, $(x,y)\in D$,

Negative Semidefinite: $V(0,0)=0$, $V(x,y)\leq 0$, $(x,y)\in D$.

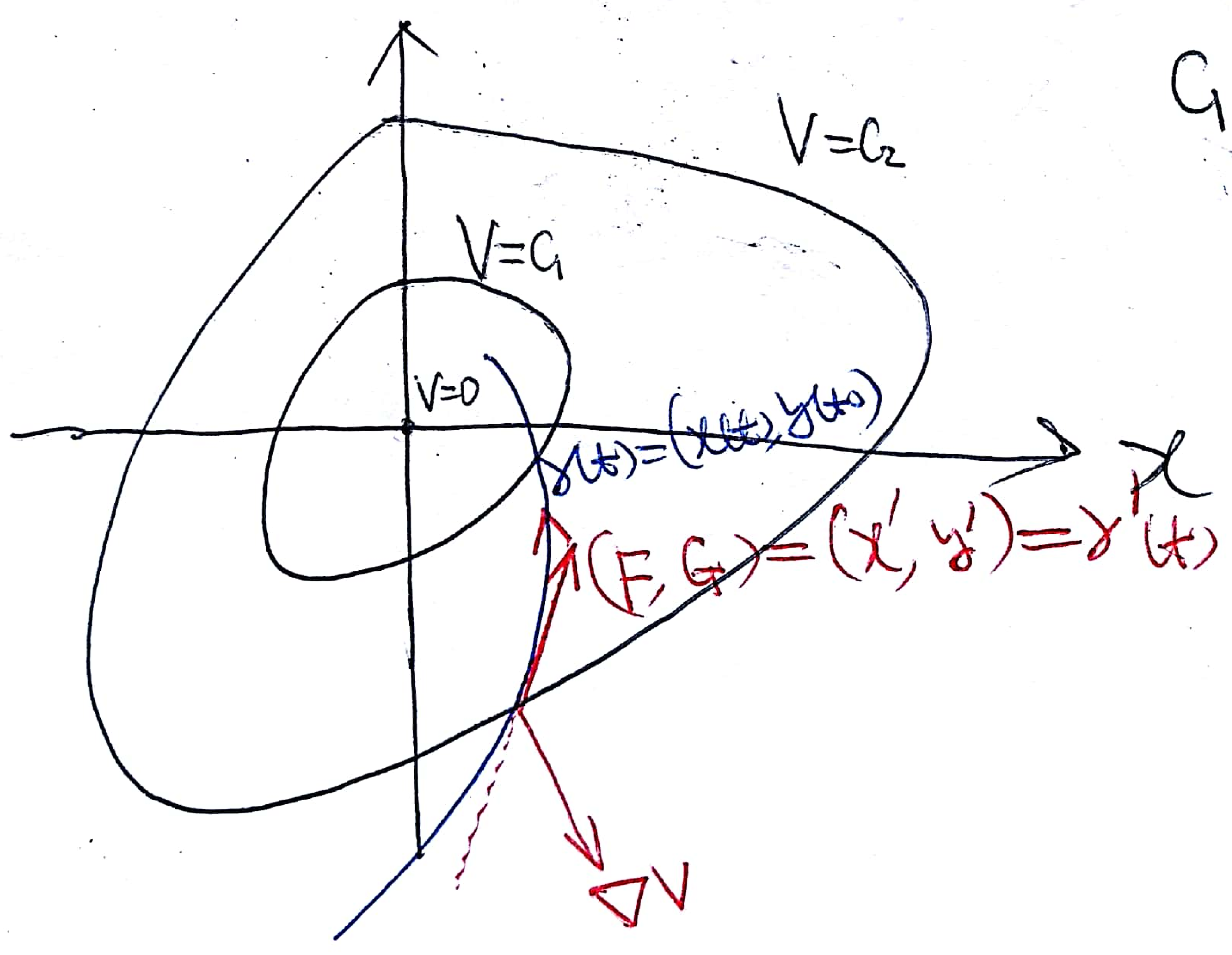
Thm 9.6.1 (Asymptotically Stable/Stable) Suppose $(0,0)$ is an isolate critical point to (GS). Assume V satisfies

(i) $V, \nabla V$ continuous near $(0,0)$;

(ii) V is positive definite near $(0,0)$;

(iii) $\partial_x V F + \partial_y V G$ negative definite near $(0,0)$
(negative semidefinite)

Then $(0,0)$ is asymptotically stable (stable).



<u>In ~~Ex~~ 2</u> $V(x,y) = 1 - \cos x + \frac{y^2}{2}$ (positive definite near $(0,0)$)

$F = y, \quad G = -\sin x$

$\nabla V = (\sin x, \; y)$

$F V_x + G V_y = y \sin x - \sin x \, y = 0$ (positive semidefinite) ≥ 0.

By Thm 9.6.1 $\Longrightarrow$ $(0,0)$ stable!!

Thm 9.6.2 (Unstable) Suppose $(0,0)$ is an isolated critical point to (GS). Assume V satisfies

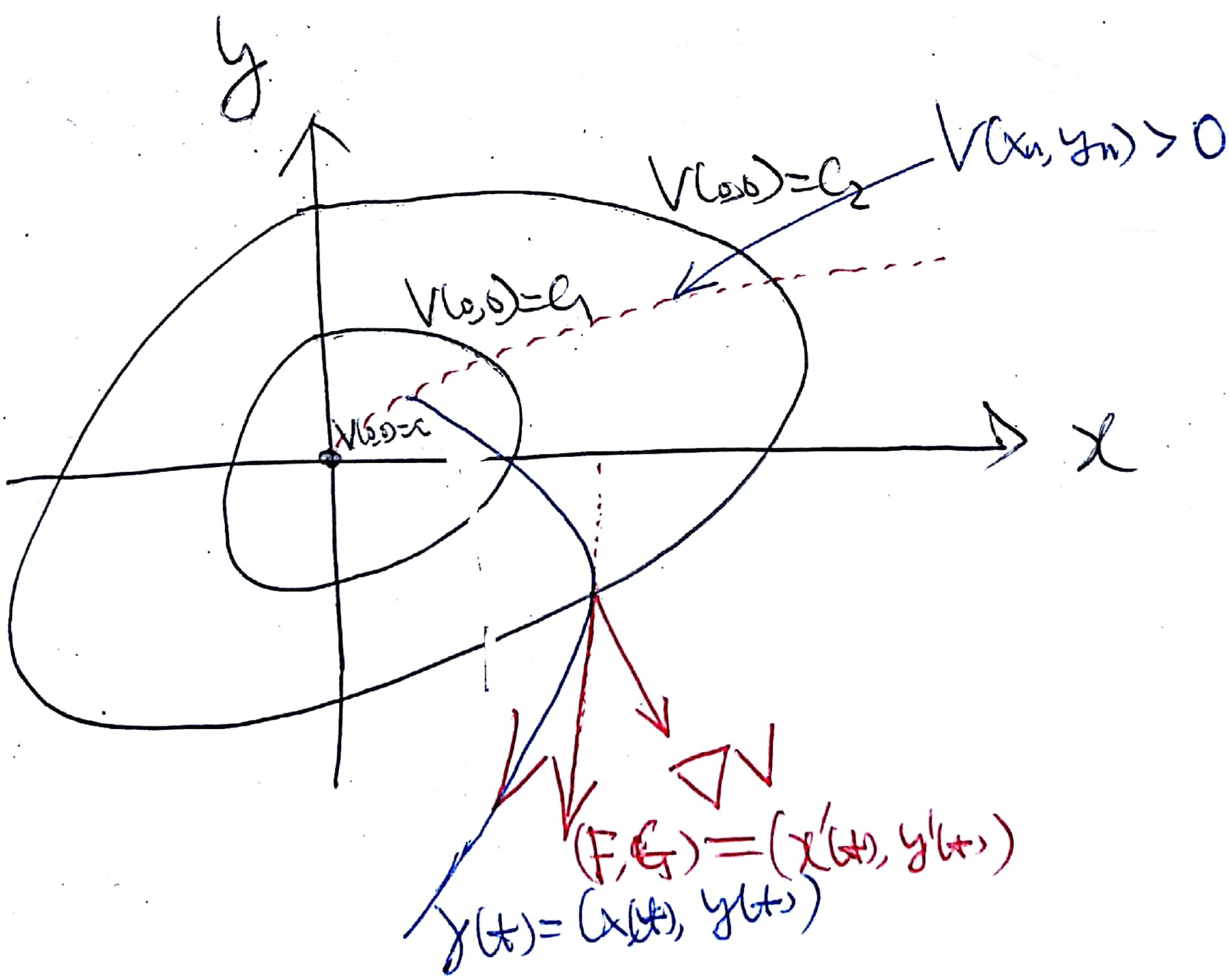
(i) $V(0,0)=0$,

(ii) $\exists\ (x_n,y_n)\to 0$, s.t. $V(x_n,y_n)>0$ (<0)

(iii) $\partial_x V F + \partial_y V G$ positive definite

(negative definite)

Then $(0,0)$ is unstable



$V(x,y)$ in Thm 9.6.1, Thm 9.6.2 are called Lyapunov function; the method of using Lyapunov function to determine stability of (C.S) by Thm 9.6.1, Thm 9.6.2 is called Lyapunov's second method.