7.5 Homogeneous Linear Systems with Constant Coefficients

Let $A = A_{n\times n}$ is a constant matrix. Consider

$$\vec{x}'(t) = A\vec{x}(t).$$

Eigenvalues of A: $\lambda_1, \lambda_2, \cdots, \lambda_k$,

Algebraic Multiplicities: $m_1, m_2, \cdots, m_k$

Geometric Multiplicities: $g_1, g_2, \cdots, g_k$

$$1 \le g_i \le m_i, \quad i = 1, 2, \cdots, k$$

$$m_1 + m_2 + \cdots + m_k = n.$$

Expect: find m_i linearly independent solutions corresponding to λ_i.

λ: eigenvalue

m: Algebraic multiplicity

g: Geometric multiplicity

$\vec{r}_1, \vec{r}_2, \dots, \vec{r}_g$ eigenvectors (linearly independent)

$A\vec{r}_i = \lambda\vec{r}_i$

Then

$$\frac{d}{dt}\left(\vec{r}_i e^{\lambda t}\right) = \lambda \vec{r}_i e^{\lambda t} = A\vec{r}_i e^{\lambda t}.$$

$\Rightarrow$ g linearly independent solutions

$$\{\vec{r}_1 e^{\lambda t}, \vec{r}_2 e^{\lambda t}, \dots, \vec{r}_g e^{\lambda t}\}.$$

<u>Conclusion</u> If Geometric Multiplicity = Algebraic Multiplicity, then m linearly independent solutions.

Ex 1. Find the general solution of

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \vec{x}.$$

Sol. Characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3$$

$$= (\lambda-3)(\lambda+1) = 0,$$

obtaining

$$\lambda_1 = 3, \quad \lambda_2 = -1.$$

For $\lambda_1 = 3$, $\quad \vec{r}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

For $\lambda_2 = -1$, $\quad \vec{r}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

The corresponding solutions

$$\vec{x}^{(1)}(t) = \vec{r}_1 e^{\lambda_1 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t},$$

$$\vec{x}^{(2)}(t) = \vec{r}_2 e^{\lambda_2 t} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}.$$

The Wronskian of these solutions is

$$W(\vec{x}^{(1)}, \vec{x}^{(2)})(t) = \begin{vmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{vmatrix} = -4e^{2t} \neq 0.$$

Hence $\{\vec{x}^{(1)}, \vec{x}^{(2)}\}$ is a fundamental set of solutions, and the general solution is

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}, \quad c_1, c_2 \in \mathbb{R}. \quad \#$$

Ex 2 Find the general solution of

$$\vec{x}' = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix} \vec{x}$$

Sol. The characteristic equation

$$\begin{vmatrix} -1-\lambda & 2 \\ -2 & -1-\lambda \end{vmatrix} = (\lambda+1)^2 + 4 = 0,$$

obtaining

$$\lambda_1 = -1+2i, \quad \lambda_2 = -1-2i.$$

For $\lambda_1 = -1+2i$, the corresponding eigenvector

$$\vec{r}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

For $\lambda_2 = -1-2i$, the corresponding eigenvector

$$\vec{r}_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

The corresponding solutions

$$\vec{x}^{(1)}(t) = \vec{r}_1 e^{\lambda_1 t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-1+2i)t}$$

$$= \begin{pmatrix} 1 \\ i \end{pmatrix} e^{-t} (\cos 2t + i \sin 2t)$$

$$= \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-t} + i \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-t},$$

$$\vec{x}^{(2)}(t) = \vec{r}_2 e^{\lambda_2 t} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{(-1-2i)t}$$

$$= \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-t} (\cos 2t - i \sin 2t)$$

$$= \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-t} + i \begin{pmatrix} -\sin 2t \\ -\cos 2t \end{pmatrix} e^{-t},$$

which are complex-valued solutions. Taking the real and imaginary parts leads to two real-valued solutions

$$\vec{u}(t) = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-t}, \quad \vec{v}(t) = \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-t}.$$

The Wronskian is

$$W(u,v)(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{vmatrix} e^{-2t} = e^{-2t} \neq 0.$$

Therefore $\{\vec{u}, \vec{v}\}$ is a fundamental set of solutions.

The general solution is

$$\vec{x}(t) = C_1 \vec{u}(t) + C_2 \vec{v}(t),$$

$$= C_1 \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} e^{-t} + C_2 \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix} e^{-t}, \quad C_1, C_2 \in \mathbb{R}$$

Repeated Eigenvalue

$$y''' - 3y'' + 3y' - y = 0 \qquad \text{(ODE)}$$

$x_1 = y$

$x_2 = y'$

$x_3 = y''$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \vec{x}' = \begin{pmatrix} x_1' \\ x_2' \\ x_3' \end{pmatrix} = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix}$$

$$\Rightarrow \vec{x}' = \begin{pmatrix} x_2 \\ x_3 \\ y''' \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ 3y'' - 3y' + y \end{pmatrix} = \begin{pmatrix} x_2 \\ x_3 \\ 3x_3 - 3x_2 + x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

ie

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{pmatrix} \vec{x}. \qquad \text{(HS)}$$

① Characteristic equation for (ODE)

$$\lambda^3-3\lambda^2+3\lambda-1=(\lambda-1)^3=0$$

$\Rightarrow$ $\lambda=1$, multiplicity 3

$\Rightarrow$ $y_1=e^{\lambda t}$, $y_2=te^{\lambda t}$, $y_3=t^2e^{\lambda t}$

②

$$\vec{x}^{(1)}=\begin{pmatrix} y_1 \\ y_1' \\ y_1'' \end{pmatrix}=\begin{pmatrix} e^{\lambda t} \\ \lambda e^{\lambda t} \\ \lambda^2 e^{\lambda t} \end{pmatrix}=\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}e^{\lambda t}$$

$$\vec{x}^{(2)}=\begin{pmatrix} y_2 \\ y_2' \\ y_2'' \end{pmatrix}=\begin{pmatrix} te^{\lambda t} \\ (1+\lambda t)e^{\lambda t} \\ (2\lambda+\lambda^2 t)e^{\lambda t} \end{pmatrix}=\left[\begin{pmatrix} 0 \\ 1 \\ 2\lambda \end{pmatrix}+\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}t\right]e^{\lambda t}$$

$$\vec{x}^{(3)}=\begin{pmatrix} y_3 \\ y_3' \\ y_3'' \end{pmatrix}=\begin{pmatrix} t^2e^{\lambda t} \\ (2t+\lambda t^2)e^{\lambda t} \\ (2+4\lambda t+\lambda^2 t^2)e^{\lambda t} \end{pmatrix}$$

$$=\left[\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}+\begin{pmatrix} 0 \\ 2 \\ 4\lambda \end{pmatrix}t+\begin{pmatrix} 1 \\ \lambda \\ \lambda^2 \end{pmatrix}t^2\right]e^{\lambda t}$$

Repeated Eigenvalue.

λ: eigenvalue

geometric multiplicity < algebraic multiplicity

$\vec{r}$: eigenvector $A\vec{r} = \lambda\vec{r}$, $\vec{r} \neq 0$

$\Rightarrow$

$\vec{r}e^{\lambda t}$ is a solution.

Try another solution of the form $\vec{y} = t\vec{r}e^{\lambda t}$

$$\vec{y}' - A\vec{y} = (\vec{r} + \lambda t\vec{r})e^{\lambda t} - tA\vec{r}e^{\lambda t}$$
$$= \vec{r}e^{\lambda t} - t(A - \lambda I)\vec{r}e^{\lambda t}$$
$$= \vec{r}e^{\lambda t} \boxed{\neq 0}$$ fails!!!

Try solution of the form

$$\vec{y} = (\vec{r}_0 + t\vec{r})e^{\lambda t}$$

add the lower order term

Then

$$\vec{y}' - A\vec{y} = \left(\vec{r} + \lambda(\vec{r}_0 + \vec{r}t)\right)e^{\lambda t} - A(\vec{r}_0 + t\vec{r})e^{\lambda t}$$

$$= \left(\vec{r} + \lambda\vec{r_0} - A\vec{r_0} + (\lambda I - A)\vec{r}t\right)e^{\lambda t}$$

$$= \left[\vec{r} - (A - \lambda I)\vec{r_0}\right]e^{\lambda t} \left(= 0\right)$$

We need

$$\boxed{(A - \lambda I)\vec{r_0} = \vec{r}.}$$

(Recalling $(A-\lambda I)\vec{r} = 0$, we get

$$(A - \lambda I)^2 \vec{r_0} = 0$$

$\nwarrow$ generalized eigenvector)

<u>Ex3</u> Find general solution

$$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} \vec{x}.$$

<u>Sol.</u> Characteristic equation

$$\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0,$$

obtaining

$$\lambda_1 = \lambda_2 = 2.$$

Eigenvector $\vec{r} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$

$$(A - \lambda_1 I)\vec{r} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Obtaining $\xi_1 + \xi_2 = 0$, we can choose $\xi_1 = 1$, $\xi_2 = -1$, and

$$\vec{r} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

One solution

$$\vec{x}^{(1)} = \vec{r} e^{\lambda_1 t} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}.$$

The generalized eigenvector $\vec{r}_0 = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ satisfies

$$(A - \lambda_1 I)\vec{r}_0 = \vec{r}, \text{ ie } \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

Obtaining $\xi_1 + \xi_2 = -1$, we can choose $\xi_1 = 0$, $\xi_2 = -1$, and

$$\vec{r}_0 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Therefore, we have another solution

$$\vec{x}^{(2)} = (\vec{r}_0 + t\vec{r}) e^{\lambda_1 t} = \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] e^{2t}.$$

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The Wronskian

$$W(\vec{x}^{(1)}, \vec{x}^{(2)})(t) = \begin{vmatrix} 1 & t \\ -1 & 1-t \end{vmatrix} e^{4t}$$

$$= -e^{4t} \neq 0.$$

Therefore

$$\left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}, \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] e^{2t} \right\}$$

is a fundamental set of solutions. The general sol is

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2 \left[\begin{pmatrix} 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] e^{2t}, \quad c_1, c_2 \in \mathbb{R} \ \#.$$

● Generally

λ: eigenvalue

m: algebraic multiplicity

① If one can find m linearly independent eigenvectors $r_1, \dots, r_m$, then obtain m linearly independent solutions $r_i e^{\lambda t}$, $i = 1, \dots, m$.

② Otherwise, look for solutions of the form

$$\vec{y}(t) = \left(\vec{r}_0 + \vec{r}_1 t + \vec{r}_2 \frac{t^2}{2!} + \cdots + \vec{r}_{m-1} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t}$$

Then

$$\vec{y}'(t) - A\vec{y}$$

$$= \left(\vec{r}_1 + \vec{r}_2 t + \vec{r}_3 \frac{t^2}{2!} + \cdots + \vec{r}_{m-1} \frac{t^{m-2}}{(m-2)!}\right) e^{\lambda t}$$

$$+ \left(\lambda\vec{r}_0 + \lambda\vec{r}_1 t + \lambda\vec{r}_2 \frac{t^2}{2!} + \cdots + \lambda\vec{r}_{m-1} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t}$$

$$- \left(A\vec{r}_0 + A\vec{r}_1 t + A\vec{r}_2 \frac{t^2}{2!} + \cdots + A\vec{r}_{m-1} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t}$$

$$= \vec{r}_1 - (A-\lambda I)\vec{r}_0 e^{\lambda t}$$

$$+ \left[\vec{r}_2 - (A-\lambda I)\vec{r}_1\right] t e^{\lambda t}$$

$$+ \left[\vec{r}_3 - (A-\lambda I)\vec{r}_2\right] \frac{t^2}{2!} e^{\lambda t}$$

$$+ \cdots$$

$$+ \left[\vec{r}_{m-1} - (A-\lambda I)\vec{r}_{m-2}\right] \frac{t^{m-2}}{(m-2)!} e^{\lambda t}$$

$$- (A-\lambda I)\vec{r}_{m-1} \frac{t^{m-1}}{(m-1)!} e^{\lambda t}$$

11.

In order that $\vec{y}' = A\vec{y}$, we need

$$\begin{cases} (A-\lambda I)\vec{r}_0 = \vec{r}_1 \\ (A-\lambda I)\vec{r}_1 = \vec{r}_2 & \rightsquigarrow \vec{r}_2 = (A-\lambda I)^2 \vec{r}_0 \\ \vdots \\ (A-\lambda I)\vec{r}_{m-2} = \vec{r}_{m-1} & \rightsquigarrow \vec{r}_{m-1} = (A-\lambda I)^{m-1}\vec{r}_0 \\ (A-\lambda I)\vec{r}_{m-1} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} (A-\lambda I)^m \vec{r}_0 = 0 \\ \vec{r}_1 = (A-\lambda I)\vec{r}_0 \\ \vec{r}_2 = (A-\lambda I)\vec{r}_1 \\ \vdots \\ \vec{r}_{m-1} = (A-\lambda I)\vec{r}_{m-2} \end{cases} \qquad (*)$$

☆ Generally if λ is an eigenvalue of A with algebraic multiplicity m, then

$$(A-\lambda I)^m \vec{r} = 0$$

has m linearly independent solutions

$$\vec{r}_0^{(1)}, \quad \vec{r}_0^{(2)}, \quad \cdots, \quad \vec{r}_0^{(m)}$$

Correspondingly, by (*), we have

$$\vec{r}_1^{(1)} = (A-\lambda I)\vec{r}_0^{(1)}, \quad \vec{r}_1^{(2)} = (A-\lambda I)\vec{r}_0^{(2)}, \quad \cdots \quad \vec{r}_1^{(m)} = (A-\lambda I)\vec{r}_0^{(m)}$$

$$\vec{r}_2^{(1)} = (A-\lambda I)\vec{r}_1^{(1)}, \quad r_2^{(2)} = (A-\lambda I) r_1^{(2)}, \quad \cdots \quad r_2^{(m)} = (A-\lambda I) r_1^{(m)}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\vec{r}_{m-1}^{(1)} = (A-\lambda I) r_{m-2}^{(1)}, \quad r_{m-1}^{(2)} = (A-\lambda I) r_{m-2}^{(2)}, \quad \cdots \quad r_{m-1}^{(m)} = (A-\lambda I) r_{m-2}^{(m)}$$

And m linearly independent solutions

$$\begin{cases} \vec{x}^{(1)}(t) = \left(\vec{r}_0^{(1)} + \vec{r}_1^{(1)} t + \vec{r}_2^{(1)} \frac{t^2}{2!} + \cdots + \vec{r}_{m-1}^{(1)} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t} \\ \vec{x}^{(2)}(t) = \left(\vec{r}_0^{(2)} + \vec{r}_1^{(2)} t + \vec{r}_2^{(2)} \frac{t^2}{2!} + \cdots + \vec{r}_{m-1}^{(2)} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t} \\ \vdots \\ \vec{x}^{(m)}(t) = \left(\vec{r}_0^{(m)} + \vec{r}_1^{(m)} t + \vec{r}_2^{(m)} \frac{t^2}{2!} + \cdots + \vec{r}_{m-1}^{(m)} \frac{t^{m-1}}{(m-1)!}\right) e^{\lambda t} \end{cases}$$

Ex 4 Find the general solution of

$$\vec{x}' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \vec{x}.$$

Sol. Characteristic equation

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2-2\lambda+1) = -(\lambda-1)^3 = 0,$$

obtaining

$$\lambda_1 = \lambda_2 = \lambda_3 = 1.$$

Then

$$(A-\lambda I) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$(A-\lambda I)^2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(A-\lambda I)^3 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

System

$$(A-\lambda I)^3 \vec{r}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{r}_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

has three linearly independent solutions

$$\vec{r}_0^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{r}_0^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{r}_0^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Set $\vec{r}_1^{(1)}, \vec{r}_1^{(2)}, \vec{r}_1^{(3)}$ as

$$\vec{r}_1^{(1)} = (A-\lambda I)\vec{r}_0^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{r}_1^{(2)} = (A-\lambda I)\vec{r}_0^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

$$\vec{r}_1^{(3)} = (A-\lambda I)\vec{r}_0^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

And $\vec{r}_2^{(1)}, \vec{r}_2^{(2)}, \vec{r}_2^{(3)}$ as

$$\vec{r}_2^{(1)} = (A-\lambda I)\vec{r}_1^{(1)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{r}_2^{(2)} = (A-\lambda I)\vec{r}_1^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{r}_2^{(3)} = (A-\lambda I)\vec{r}_1^{(3)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Hence, we have three solutions

$$\vec{x}^{(1)} = \left(\vec{r}_0^{(1)} + \vec{r}_1^{(1)} t + \vec{r}_2^{(1)} \frac{t^2}{2!}\right) e^{\lambda t}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} = \begin{pmatrix} e^t \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{x}^{(2)} = \left(\vec{r}_0^{(2)} + \vec{r}_1^{(2)} t + \vec{r}_2^{(2)} \frac{t^2}{2!}\right) e^{\lambda t}$$

$$= \left[\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2!}\right] e^{t}$$

$$= \begin{pmatrix} t - \frac{t^2}{2} \\ 1-t \\ -t \end{pmatrix} e^{t}$$

$$\vec{x}^{(3)} = \left(\vec{r}_0^{(3)} + \vec{r}_1^{(3)} t + \vec{r}_2^{(3)} \frac{t^2}{2!}\right) e^{\lambda t}$$

$$= \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \frac{t^2}{2}\right] e^{t}$$

$$= \begin{pmatrix} \frac{t^2}{2} \\ t \\ 1+t \end{pmatrix} e^{t}$$

The Wronskian

$$W\left(\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}\right)(t) = \begin{vmatrix} 1 & t-\frac{t^2}{2} & \frac{t^2}{2} \\ 0 & 1-t & t \\ 0 & -t & 1+t \end{vmatrix} e^{3t}$$

$$= e^{3t} \neq 0$$

Hence $\{\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}\}$ is a fundamental set of solutions. The general solution is

$$\vec{x}(t) = c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t) + c_3 \vec{x}^{(3)}(t)$$

$$= c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{t} + c_2 \begin{pmatrix} t-\frac{t^2}{2} \\ 1-t \\ -t \end{pmatrix} e^{t} + c_3 \begin{pmatrix} \frac{t^2}{2} \\ t \\ 1+t \end{pmatrix} e^{t},$$

$c_1, c_2, c_3 \in \mathbb{R}$. #

Ex. Find the general solution of

$$\vec{x}' = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 2 & 1 \\ -1 & 0 & 2 \end{pmatrix} \vec{x}.$$

Sol. Characteristic equation

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ -2 & 2-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda)^2 = 0$$

obtaining

$$\lambda_1 = 1, \quad \lambda_2 = \lambda_3 = 2.$$

Eigenvector for $\lambda_1 = 1$:

$$\begin{pmatrix} 0 & 0 & 0 \\ -2 & 1 & 1 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \xi_1 = \xi_2 = \xi_3$, choosing

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$ one solution

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^t$$

Solutions related to $\lambda_2=\lambda_3=2$ (We use two approaches)

Approach 1 eigenvector for $\lambda_2=2$

$$(A-\lambda_2 I)\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_2\end{pmatrix}=\begin{pmatrix}-1 & 0 & 0\\ -2 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}=0$$

$\Rightarrow \xi_1=\xi_3=0$, we can choose

$$\vec{\xi}=\begin{pmatrix}0\\1\\0\end{pmatrix} \quad \text{eigenvector for } \lambda_2=2.$$

Solve $(A-\lambda_2 I)\vec{\eta}=\vec{\xi}$, i.e.

$$(A-\lambda_2 I)\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}=\begin{pmatrix}-1 & 0 & 0\\ -2 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$\Rightarrow \eta_1=0,\ \eta_3=1$, we can choose

$$\vec{\eta}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Then we have another two solutions

$$\vec{x}^{(2)}=\vec{\xi}e^{\lambda_2 t}=\begin{pmatrix}0\\1\\0\end{pmatrix}e^{2t}$$

$$\vec{x}^{(3)}=(\vec{\eta}+\vec{\xi}t)e^{\lambda_2 t}=\left[\begin{pmatrix}0\\0\\1\end{pmatrix}+t\begin{pmatrix}0\\1\\0\end{pmatrix}\right]e^{2t}$$

$$=\begin{pmatrix}0\\t\\1\end{pmatrix}e^{2t}$$

Approach 2

$$A-\lambda_2 I=\begin{pmatrix}-1 & 0 & 0\\ -2 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}$$

$$(A-\lambda_2 I)^2=\begin{pmatrix}-1 & 0 & 0\\ -2 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ -2 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}$$

System $(A-\lambda_2 I)^2\vec{r}_0=0$, i.e.

$$\begin{pmatrix}1 & 0 & 0\\ 1 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\vec{r}_0=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

has two linearly independent solutions

$$\vec{r}_0^{(1)}=\begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \vec{r}_0^{(2)}=\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

Set $\vec{r}_1^{(1)}$ and $\vec{r}_1^{(2)}$ as

$$\vec{r}_1^{(1)}=(A-\lambda_2 I)\vec{r}_0^{(1)}=\begin{pmatrix}-1 & 0 & 0\\ -2 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\1\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\vec{r}_1^{(2)}=(A-\lambda_2 I)\vec{r}_0^{(2)}=\begin{pmatrix}-1 & 0 & 0\\ -2 & 0 & 1\\ -1 & 0 & 0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}=\begin{pmatrix}0\\1\\0\end{pmatrix}$$

Then, we have another two solutions

$$\tilde{x}^{(1)} = (\vec{r}_0^{(1)} + t\vec{r}_1^{(1)}) e^{\lambda_2 t} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t} \left(= \vec{x}^{(2)}\right)$$

$$\tilde{x}^{(2)} = (\vec{r}_0^{(2)} + t r_1^{(2)}) e^{\lambda_2 t}$$

$$= \left[\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t\right] e^{2t} = \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} e^{2t} = \left(\vec{x}^{(3)}\right)$$

Therefore, we have three solutions

$$\vec{x}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{t}, \quad \vec{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} e^{2t}, \quad \vec{x}^{(3)} = \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} e^{2t}.$$

The Wronskian

$$W\left(\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}\right)(t) = \begin{vmatrix} e^{t} & 0 & 0 \\ e^{t} & e^{2t} & te^{2t} \\ e^{t} & 0 & e^{2t} \end{vmatrix}$$

$$= e^{5t} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & t \\ 1 & 0 & 1 \end{vmatrix} = e^{5t} \neq 0.$$

$\Rightarrow \{\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)}\}$ a FSS.

$\Rightarrow$ general solution

$$\vec{x} = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + c_3\vec{x}^{(3)}$$

$$= c_1\begin{pmatrix}1\\1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix}e^{2t} + c_3\begin{pmatrix}0\\t\\1\end{pmatrix}e^{2t},$$

$c_1, c_2, c_3 \in \mathbb{R}$. #