3.6 Variation of Parameters

Given $\{y_1, y_2\}$ of

$$y'' + p(t)y' + q(t)y = 0. \qquad (H)$$

Consider

$$y'' + p(t)y' + q(t)y = g(t). \qquad (NH)$$

$\{y_1, y_2\}$ of (H)

$$\Big\downarrow \; y = u_1(t)y_1 + u_2(t)y_2$$

A particular solution of (NH)

Variation of Parameters: $\left(\{y_1, y_2\} \text{ of } (H) \to y \text{ of } (NH)\right)$

Assume

$$y = u_1(t)y_1 + u_2(t)y_2. \qquad (0)$$

$$\Rightarrow \quad y' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

Let

$$y_1 u_1' + y_2 u_2' = 0 \qquad (1)$$

Then

$$y' = y_1' u_1 + y_2' u_2 \quad (2)$$

$\Rightarrow$

$$y'' = y_1'' u_1 + y_2'' u_2 + y_1' u_1' + y_2' u_2' \quad (3)$$

(1), (2), (3) $\longrightarrow$ (NH)

$\Rightarrow$

$$y_1'' u_1 + y_2'' u_2 + y_1' u_1' + y_2' u_2'$$
$$+ p(t)\left(y_1' u_1 + y_2' u_2\right) + q(t)\left(y_1 u_1 + y_2 u_2\right) = g(t)$$

i.e

$$\left(y_1'' + p(t) y_1' + q(t) y_1\right) u_1 \ (=0)$$
$$+ \left(y_2'' + p(t) y_2' + q(t) y_2\right) u_2 \ (=0)$$
$$+ y_1' u_1' + y_2' u_2' = g(t)$$

y_1, y_2 satisfies (H)

$\Rightarrow$

$$\begin{cases} y_1' u_1' + y_2' u_2' = g(t) \\ y_1 u_1' + y_2 u_2' = 0 \leftarrow (1) \end{cases}$$

$\Longrightarrow$

$$\begin{cases} u_1' = -\dfrac{y_2 g}{W} \\ u_2' = \dfrac{y_1 g}{W} \end{cases} \qquad W = W(y_1, y_2)(t)$$

$\Longrightarrow$

$$\begin{cases} u_1 = -\displaystyle\int_{t_0}^{t} \frac{y_2 g}{W}\, ds \\ u_2 = \displaystyle\int_{t_0}^{t} \frac{y_1 g}{W}\, ds \end{cases}$$

$\Longrightarrow$

$$y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s) g(s)}{W(s)}\, ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s) g(s)}{W(s)}\, ds$$

Thm (Variation of Parameters) Suppose $\{y_1, y_2\}$ is a fundamental set of solutions of

$$y'' + p(t) y' + q(t) y = 0, \qquad (H)$$

on an open interval I, where p, q are continuous on I.

Then

$$y'' + p(t) y' + q(t) y = g(t)$$

has a particular solution

$$Y(t) = -y_1(t)\int_{t_0}^{t} \frac{y_2(s) g(s)}{W(s)}\, ds + y_2(t)\int_{t_0}^{t} \frac{y_1(s) g(s)}{W(s)}\, ds$$

where $t_0 \in I$ is arbitrary, and $W(t) = W(y_1, y_2)(t)$, and the general solution of (NH) is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

$$= c_1 y_1(t) + c_2 y_2(t) - y_1(t) \int_{t_0}^{t} \frac{y_2(s) g(s)}{W(s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s) g(s)}{W(s)} ds.$$

Reduction of Order (y_1 of (H) $\to$ y of (NH))

y_1: Solution of (H)

$y = u(t) y_1(t)$: Solution of (NH)

$\Longrightarrow$

$$y' = u' y_1 + u y_1'$$

$$y'' = u'' y_1 + 2u' y_1' + u y_1''$$

$\Longrightarrow$

$$g(t) = y'' + p y' + q y$$

$$= u'' y_1 + 2u' y_1' + u y_1'' + p(u' y_1 + u y_1') + q u y_1$$

$$= y_1 u'' + (2y_1' + p y_1) u' + (y_1'' + p y_1' + q y_1) u$$

$$\Rightarrow$$

$$y_1 u'' + (2y_1' + py_1)u' = g$$

$$\Rightarrow$$

$$u'' + \left(2\frac{y_1'}{y_1} + p\right)u' = \frac{g}{y_1}$$

$$\Rightarrow$$

$$u' = e^{-\int\left(2\frac{y_1'}{y_1} + p\right)dt}\left(C_1 + \int \frac{g}{y_1} e^{\int\left(2\frac{y_1'}{y_1} + p\right)dt} dt\right)$$

$$= \frac{1}{y_1^2 e^{\int p\,dt}}\left(C_1 + \int g y_1 e^{\int p\,dt} dt\right)$$

$$\Rightarrow$$

$$u = C_1 \int \frac{dt}{y_1^2 e^{\int p\,dt}} + \int \frac{\int g y_1 e^{\int p\,dt} dt}{y_1^2 e^{\int p\,dt}} dt + C_2$$

$$\Rightarrow$$

$$y = C_1 y_1 \int \frac{dt}{y_1^2 e^{\int p\,dt}} + C_2 y_1 + y_1 \int \frac{\int g y_1 e^{\int p\,dt} dt}{y_1^2 e^{\int p\,dt}} dt$$

Remark: One can either use the formular of reduction of order, or follow the argument of reduction of order.

Ex1 (Variation of Parameters) Find a particular solution of

$$y'' + 4y = \frac{3}{\sin t}.$$

Sol. The homogeneous equation

$$y'' + 4y = 0$$

has a fundamental set of solution

$$\{y_1, y_2\} = \{\cos 2t, \sin 2t\}.$$

The Wronskian

$$W(t) = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2\cos^2 2t + 2\sin^2 2t = 2.$$

By Thm

$$Y(t) = -y_1(t)\int \frac{y_2(t)g(t)}{W(t)}\,dt + y_2(t)\int \frac{y_1(t)g(t)}{W(t)}\,dt$$

$$= -\cos 2t \int \frac{\sin 2t \, \frac{3}{\sin t}}{2}\,dt + \sin 2t \int \frac{\cos 2t\,\frac{3}{\sin t}}{2}\,dt$$

$$= -3\cos 2t \int \cos t\,dt + \frac{3}{2}\sin 2t \int \frac{\cos 2t}{\sin t}\,dt$$

$$= -3\cos 2t \sin t + \frac{3}{2}\sin 2t \int \frac{1-2\sin^2 t}{\sin t}\,dt$$

$$= -3\cos 2t \sin t + \frac{3}{2}\sin 2t \int \left(\frac{1}{\sin t} - 2\sin t\right)dt$$

$$= -3\cos 2t \sin t + 3\sin 2t \cos t + \frac{3}{2}\sin 2t \int \frac{dt}{\sin t}$$

$$= 3\sin t + \frac{3}{2}\sin 2t \int \frac{dt}{\sin t}$$

$$= 3\sin t + \frac{3}{2}\sin 2t \ln\left|tg\frac{t}{2}\right|.$$

$\Rightarrow$ A particular solution

$$Y(t) = 3\sin t + \frac{3}{2}\sin 2t \ln\left|tg\frac{t}{2}\right|$$

$\Rightarrow$ general solution

$$y(t) = Y(t) + C_1 y_1(t) + C_2 y_2(t)$$

$$= 3\sin t + \frac{3}{2}\sin 2t \ln\left|tg\frac{t}{2}\right| + C_1\cos 2t + C_2\sin 2t,$$

$$C_1, C_2 \in \mathbb{R}.$$

Ex 2 (Reduction of Order) (Sol to Homogeneous eqn)

$$t^2y''+7ty'+5y=3t, \quad t>0; \quad y_1(t)=\frac{1}{t}$$

Sol. Assume

$$y = u(t)y_1(t) = \frac{u(t)}{t}$$

$$y' = \frac{u'}{t} - \frac{u}{t^2}$$

$$y'' = \frac{u''}{t} - 2\frac{u'}{t^2} + \frac{2}{t^3}u$$

$\Longrightarrow$

$$t^2\left(\frac{u''}{t} - 2\frac{u'}{t^2} + \frac{2}{t^3}u\right) + 7t\left(\frac{u'}{t} - \frac{u}{t^2}\right) + \frac{5u}{t} = 3t$$

$$tu'' - 2u' + \cancel{\frac{2}{t}u} + 7u' - \cancel{7\frac{u}{t}} + \cancel{5\frac{u}{t}} = 3t$$

$$tu'' + 5u' = 3t \Rightarrow u'' + \frac{5}{t}u' = 3$$

$\Longrightarrow$

$$u' = e^{-\int \frac{5}{t}dt}\left(c_1 + 3\int e^{\int \frac{5}{t}dt} dt\right)$$

$$= \frac{1}{t^5}\left(c_1 + \frac{t^6}{2}\right)$$

$\Rightarrow$

$$u = C_2 + C_1\int \frac{dt}{t^5} + \int \frac{t}{2}\,dt$$

$$= \frac{C_1}{t^4} + C_2 + \frac{t^2}{4}$$

$\Rightarrow$

$$y = uy_1 = \frac{C_1}{t^5} + \frac{C_2}{t} + \frac{t}{4}, \quad C_1, C_2 \in \mathbb{R}.$$

Remark: The equation in Ex 2 is an Euler equation, the particular solution $y_1 = \frac{1}{t}$ can be found explicitly.

4.1 General Theory of nth order linear ODE

- general form

$$P_0(t)y^{(n)} + P_1(t)y^{(n-1)} + \cdots + P_{n-1}(t)y' + P_n(t)y = G(t)$$

- Standard form

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_{n-1}(t)y' + p_n(t)y = g(t)$$

Parallel to the 2^{nd} order case, we have

① Existence and Uniqueness

② Defs of Fundamental Set of Solutions
General Solution

③ Existence of Fundamental Set of Solutions

Thm 4.1.1 ($\exists !$)

$I: \alpha < t < \beta$, $t_0 \in I$

$g, p_i, i = 1, 2, \cdots, n$ Continuous on I (Assm)

$\Rightarrow \exists !$ Solution $y = \phi(t)$ to IVP

$$\begin{cases} y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = g(t) \\ y(t_0) = y_0,\ y'(t_0) = y_0',\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)} \end{cases}$$

On I.

(I) Homogeneous Equations

Wronskian

$$W(y_1, \cdots, y_n)(t) = \begin{vmatrix} y_1(t) & y_2(t) & \cdots & y_n(t) \\ y_1'(t) & y_2'(t) & \cdots & y_n'(t) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \cdots & y_n^{(n-1)}(t) \end{vmatrix}$$

Homogeneous Equation

$$y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = 0 \qquad (H)$$

Def ≪Fundamental Set of Solutions≫

$\{y_1, \cdots, y_n\}$ is called FSS of (H), if

(i) y_i, $i=1,\cdots,n$ are solutions of H on I

& (ii) $W(y_1,\cdots,y_n)(t_0) \neq 0$, for some $t_0 \in I$

Def ≪General Solution of (H)≫

$\{y_1, \cdots, y_n\}$ a FSS of (H) on I

$y = c_1y_1 + \cdots + c_ny_n$, $c_1, \cdots, c_n \in \mathbb{R}$ general solution.

Thm 4.1.2 Assume (Assm) holds, $\{y_1, \cdots, y_n\}$ a FSS of (H) on I. Then every solution y of (H) on I can be expressed as

$$y = c_1y_1 + \cdots + c_ny_n, \quad c_1, \cdots, c_n \in \mathbb{R}.$$

Remark: Under (Assm), the FSS of (H) on I always exists

- Linear Dependence and Independence

$f_1, f_2, \cdots, f_n$ defined on $I: \alpha < t < \beta$

<u>linearly dependent</u>: $\exists\, k_1, k_2, \cdots, k_n$ not all zero, st

$$k_1 f_1(t) + \cdots + k_n f_n(t) = 0, \quad \forall t \in I.$$

<u>linearly Independent</u> = not linearly Independent.

<u>Ex3</u> Linearly dependent or Independent?

(a) $f_1(t) = 1, \quad f_2(t) = t, \quad f_3(t) = t^2$, on $(-\infty, \infty)$

(b) $f_1(t) = t|t|, \quad f_2(t) = t^2$ on $(-\infty, \infty)$.

<u>Sol.</u> (a) Suppose k_1, k_2, k_3, st

$$k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = 0, \quad \forall t \in \mathbb{R},$$

i.e.

$$k_1 + k_2 t + k_3 t^2 = 0, \quad \forall t \in \mathbb{R}.$$

choosing $t=0$, $t=-1$, $t=1$ yields

$$\begin{cases} k_1 = 0 \\ k_1 - k_2 + k_3 = 0 \\ k_1 + k_2 + k_3 = 0 \end{cases} \Rightarrow k_1 = k_2 = k_3 = 0.$$

Therefore $1, t, t^2$ are linearly independent on $\mathbb{R}$.

(b) Suppose

$$k_1 |t| t + k_2 t^2 = 0, \quad \forall t \in \mathbb{R}$$

Choosing $t=1$, $t=-1$ yields

$$\begin{cases} k_1 + k_2 = 0 \\ -k_1 + k_2 = 0 \end{cases} \Rightarrow k_1 = k_2 = 0.$$

Therefore $|t|t, t^2$ are linearly independent on $\mathbb{R}$ #

Thm 4.1.3 Assume (Assm) holds. $\{y_1, \cdots, y_n\}$ solution of (H). Then

$\{y_1, \cdots, y_n\}$ FSS of (H) on I

$\Leftrightarrow$ $y_1, \cdots, y_n$ linearly independent on I.

Pf $\Rightarrow$ Assume $k_1, k_2, \cdots, k_n$, s.t.

$$k_1 y_1(t) + k_2 y_2(t) + \cdots + k_n y_n(t) = 0, \quad \forall t \in I$$

then

$$k_1 y_1^{(i)}(t) + k_2 y_2^{(i)}(t) + \cdots + k_n y_n^{(i)}(t) = 0, \quad \forall t \in I, \ i = 1, 2, \cdots, n-1.$$

In particular

$$k_1 y_1^{(i)}(t_0) + \cdots + k_n y_n^{(i)}(t_0) = 0, \quad i = 0, 1, \cdots, n-1,$$

that is

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

But, since $\{y_1, \cdots, y_n\}$ is a FSS of (H) on I, we have

$$W(y_1, \cdots, y_n)(t_0) \neq 0$$

$$\Rightarrow \quad k_1 = k_2 = \cdots = k_n = 0.$$

therefore $y_1, \cdots, y_n$ are linearly independent on I.

($\Leftarrow$) By definition, we need to check

$$W(y_1, \cdots, y_n)(t_0) \neq 0, \text{ for some } t_0 \in I.$$

Assume, by contradiction, that

$$W(y_1, \cdots, y_n)(t_0) = 0, \text{ for all } t_0 \in I.$$

That is

$$A(t_0) = \begin{pmatrix} y_1(t_0) & \cdots & y_n(t_0) \\ \vdots & & \\ y_1^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \text{ is singular,}$$

then $\exists\, k_1, \cdots, k_n$, not all zero, s.t. $A(t_0)\vec{k} = 0$,

$\vec{k} \triangleq (k_1, \cdots, k_n)^T$, that is

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & \cdots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \cdots & y_n'(t_0) \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \cdots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Set

$$y = k_1 y_1 + k_2 y_2 + \cdots + k_n y_n.$$

Then

- y is a solution of $(*)$ on I
- $y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$.

By $\exists!$, we have

$$0 = y(t) = k_1 y_1(t) + \cdots + k_n y_n(t), \quad \forall t \in I$$

Contradictory to the linearly independence of $y_1, \dots, y_n$. #

(II) Nonhomogeneous Equations

$$y^{(n)} + p_1(t) y^{(n-1)} + \dots + p_{n-1}(t) y' + p_n(t) y = g(t) \quad (NH)$$

Let

- y a solution to (NH)
- Y a particular solution to (NH)

then

$y - Y$ a solution to (H)

By Thm 4.1.2

$$y - Y = c_1 y_1 + \dots + c_n y_n$$

i.e.

$$y = Y + c_1 y_1 + \dots + c_n y_n, \quad c_1, \dots, c_n \in \mathbb{R}.$$

$\uparrow$

general solution of (NH).