3.5 Nonhomogeneous Equations: Method of Undetermined Coefficients (Continue)

$$y'' + ay' + by = g(t)$$

$\uparrow$ nonhomogeneous term.

<u>AIM:</u> Find a particular solution

<u>Simple Calculations:</u>

$$\frac{d^k}{dt^k}\{\text{Polynomials}\} \rightsquigarrow \text{Polynomials}$$

$$\frac{d^k}{dt^k}\{\text{exponential functions}\} \rightsquigarrow \{\text{exponential functions}\}$$

$$\frac{d^k}{dt^k}\{\text{sines, cosines}\} \rightsquigarrow \{\text{sines, cosines}\}.$$

$$\frac{d^k}{dt^k}\{\text{Sum, product of Polynomials, exponential functions or sines, cosines}\} \triangleq X$$

$$\rightsquigarrow X$$

$$\Longrightarrow$$

$(Lf \triangleq f'' + af' + bf)$

$$L X \to X.$$

Conclusion: If $g \in X$, i.e. g is the sum, product of functions from polynomial, exponential functions, sines or cosines, then one can expect a particular solution $y \in X$, s.t. $Ly = g$.

Ex1 Find a particular solution of

$$y'' - 3y' - 4y = 3e^{2t}.$$

Sol. Expect solutions of the form Ae^{2t}. Then

$$y'' - 3y' - 4y = A(2^2 - 3\times 2 - 4)e^{2t}$$

$$= -6Ae^{2t} = 3e^{2t}$$

We get $A = -\frac{1}{2}$, and thus find a particular solution

$$Y(t) = -\frac{1}{2}e^{2t}.$$

Ex Find a particular solution of

$$y'' - 3y' - 4y = 2\sin t$$

Sol. ① Try $Y(t) = A\sin t$. Then

$$Y'' - 3Y' - 4Y = (-A-4)\sin t - 3A\cos t$$
$$= A\sin t.$$

Need
$$\begin{cases} -A-4 = A \\ -3A = 0 \end{cases} \Rightarrow \text{No such } A!$$

$\Rightarrow$ No solution of the form $A\sin t$.

② Note that $(\sin t)' = \cos t$, $(\cos t)' = -\sin t$. One may try $Y(t) = A\sin t + B\cos t$. Then

$$Y'' - 3Y' - 4Y = (3B - 5A)\sin t - (3A + 5B)\cos t.$$
$$= 2\sin t$$

We need
$$\begin{cases} 3B - 5A = 2 \\ 3A + 5B = 0 \end{cases} \Rightarrow \begin{cases} A = -\frac{5}{17} \\ B = \frac{3}{17} \end{cases}$$

Therefore

$$Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$$

Ex Find a particular solution of

$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$

Sol. Try solution

$$Y(t) = (A\cos 2t + B\sin 2t)e^{t}.$$

Then

$$Y'(t) = (A+2B)e^{t}\cos 2t + (B-2A)e^{t}\sin 2t$$

$$Y''(t) = (4B-3A)e^{t}\cos 2t - (4A+3B)e^{t}\sin 2t$$

Substituting Y and Y'' into the P.DE $\rightarrow$

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0 \end{cases}$$

$\Rightarrow$

$$A = \frac{10}{13} \qquad B = \frac{2}{13}$$

$\Rightarrow$

$$Y(t) = \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t.$$

<u>Example</u> Find a particular solution to

$$y'' - 3y' - 4y = 2e^{-t}.$$

<u>Sol.</u> Try to find solution of form $Y(t) = Ae^{-t}$. Then

$$Y'' - 3Y' - 4Y = A(1+3-4) = 0$$
$$\neq 2e^{-t}!$$

$\Rightarrow$ No solution of the form Ae^{-t}.

<u>Check</u> $r=-1$ is a root of

$$r^2 - 3r - 4 = 0. \qquad (CE)$$

$$r_1 = -1, \ r_2 = 4$$

$$(CE) \Leftrightarrow (r+1)(r-4) = 0$$

<u>Calculate</u>

$$\left(\frac{d}{dt}+1\right)\left(\frac{d}{dt}-4\right)$$
$$= \frac{d}{dt}\left(\frac{d}{dt}-4\right) + \frac{d}{dt} - 4 = \frac{d^2}{dt^2} - 4\frac{d}{dt} + \frac{d}{dt} - 4$$
$$= \frac{d^2}{dt^2} - 3\frac{d}{dt} - 4$$

$\Rightarrow$

$$y'' - 3y' - 4y = \left(\frac{d}{dt}+1\right)\left(\frac{d}{dt}-4\right)y$$

Then

$$y'' - 3y' - 4y = 2e^{-t} \qquad \text{(ODE)}$$

$\Leftrightarrow$

$$\left(\frac{d}{dt} - 4\right)\left(\frac{d}{dt} + 1\right) y = 2e^{-t}$$

Set

$$x(t) = \left(\frac{d}{dt} + 1\right) y, \text{ i.e. } x = y' + y$$

$\Rightarrow$

$$\left(\frac{d}{dt} - 4\right) x(t) = 2e^{-t}, \text{ i.e. } x' - 4x = 2e^{-t}$$

$\Downarrow$

$$x(t) = e^{4t} \int 2e^{-t} e^{-4t}\, dt = 2e^{4t} \int e^{-5t}\, dt$$

$$= -\frac{2}{5} e^{4t} e^{-5t} = -\frac{2}{5} e^{-t}.$$

recalling $y' + y = x$

$\Rightarrow$

$$y' + y = -\frac{2}{5} e^{-t}$$

$$y = e^{-t} \left(-\frac{2}{5}\right) \int e^{t} e^{-t}\, dt$$

$$= -\frac{2}{5} t e^{-t}.$$

$\Rightarrow$

$Y(t) = -\frac{2}{5} t e^{-t}$ is a particular sol to (ODE)

More generally

$$y'' + ay' + by = P_1(t)e^{\alpha t} \qquad \text{(ODE)}$$

《Choose $P_1(t) = t$ as example》

$$r^2 + ar + b = 0 \qquad \text{(CE)}$$

r_1, r_2 roots may be repeated/complex.

Then

$$\text{(CE)} \Leftrightarrow (r - r_1)(r - r_2) = 0$$

$$\Leftrightarrow r_1 + r_2 = -a, \quad r_1 r_2 = b$$

Therefore

$$\text{(ODE)} \Leftrightarrow y'' - (r_1 + r_2)y' + r_1 r_2 y = P_1(t)e^{\alpha t}$$

Calculate

$$\left(\frac{d}{dt} - r_1\right)\left(\frac{d}{dt} - r_2\right)y$$

$$= \left(\frac{d^2}{dt^2} - (r_1 + r_2)\frac{d}{dt} + r_1 r_2\right)y$$

$$= y'' - (r_1 + r_2)y' + r_1 r_2 y$$

m: the multiplicity of α as the root of (CE)

$$\begin{cases} m = 0, & \text{if } \alpha \text{ is not a root} \\ m = 1, & \text{if } \alpha \text{ is a root but not repeat} \\ m = 2, & \text{if } \alpha \text{ is a repeated root} \end{cases}$$

Therefore

(ODE) $\Longleftrightarrow$

$$\left(\frac{d}{dt}-r_1\right)\left(\frac{d}{dt}-r_2\right)y = P_1(t)e^{\alpha t} = te^{\alpha t} \quad (\text{ODE})'$$

① $\alpha \neq r_1$, $\alpha \neq r_2$. look for solution $m=0$

$$Y(t) = Q_1(t)e^{\alpha t} = (At+B)e^{\alpha t}$$

$$\Rightarrow Y'(t) = (A+\alpha(At+B))e^{\alpha t} = (A\alpha t + A + \alpha B)e^{\alpha t}$$

$$Y''(t) = (A\alpha + \alpha(A\alpha t + A + \alpha B))e^{\alpha t}$$

$$= (A\alpha^2 t + 2A\alpha + \alpha^2 B)e^{\alpha t}$$

$$\Rightarrow Y'' + aY' + bY$$

$$= (A\alpha^2 t + 2A\alpha + \alpha^2 B + A\alpha a t + (A+\alpha B)a$$
$$+ Abt + Bb)e^{\alpha t}$$

$$= [A(\alpha^2 + a\alpha + b)t + (\alpha^2 + a\alpha + b)B$$
$$+ (2\alpha + a)A]e^{\alpha t}$$

$$= te^{\alpha t}$$

$\Rightarrow$ require

$$\begin{cases} (\alpha^2+a\alpha+b)A=1 \\ (\alpha^2+a\alpha+b)B+(2\alpha+a)A=0 \end{cases}$$

Since $\alpha\neq r_1$, $\alpha\neq r_2 \Rightarrow (\alpha^2+a\alpha+b)\neq 0$

$\Rightarrow$ A, B are solvable.

$\Rightarrow \exists$ a particular solution of the form

$$Y(t)=Q_1(t)e^{\alpha t}=t^0 R_1(t)e^{\alpha t}=t^m R_1(t)e^{\alpha t}$$

② $\alpha=r_1$, $\alpha\neq r_2$. (In this case $r_1\neq r_2$) $m=1$

$(ODE)\Leftrightarrow(ODE)'$

$$\Leftrightarrow\left(\frac{d}{dt}-r_1\right)\left(\frac{d}{dt}-r_2\right)y=P_1(t)=te^{\alpha t}$$
$$=te^{r_1 t}$$

Set

$$x(t)=\left(\frac{d}{dt}-r_2\right)y(t)$$

$\Rightarrow$

$$\left(\frac{d}{dt}-r_1\right)x=x'-r_1x=te^{r_1 t}$$

$$\Rightarrow x(t) = e^{r_1 t}\int e^{-r_1 t}\, t\, e^{r_1 t}\, dt = e^{r_1 t}\frac{t^2}{2}$$

$$\Rightarrow y' - r_2 y = \frac{t^2}{2} e^{r_1 t}$$

$$\Rightarrow y = e^{r_2 t}\frac{1}{2}\int t^2 e^{(r_1-r_2)t}\, dt$$

$$= e^{r_1 t}\left(\frac{t^2}{2(r_1-r_2)} - \frac{t}{(r_1-r_2)^2} - \frac{1}{(r_1-r_2)^3}\right)$$

$\uparrow$

a particular solution.

But, recalling $\alpha = r_1$, $\Rightarrow \dfrac{e^{r_1 t}}{(r_1-r_2)^3}$ is a solution to the homogeneous equation.

$$\Rightarrow y = e^{r_1 t}\, t\left(\frac{t}{2(r_1-r_2)} - \frac{1}{(r_1-r_2)^3}\right)$$

$$= t\, Q_1(t) e^{\alpha t} = t^m R_1(t) e^{\alpha t}$$

Is a particular solution to (ODE).

③ $\alpha = r_1 = r_2$ $\quad m = 2$

$$(ODE) \Leftrightarrow (ODE)'$$

$$\Leftrightarrow \left(\frac{d}{dt}-\alpha\right)^2 y = P_1(t) e^{\alpha t} = t e^{\alpha t}$$

Set

$$x = y' - \alpha y$$

$$\Rightarrow \quad x' - \alpha x = t e^{\alpha t}$$

$$\Rightarrow \quad x = e^{\alpha t} \int t \, dt = \frac{t^2}{2} e^{\alpha t}$$

$$\Rightarrow \quad y' - \alpha y = x = \frac{t^2}{2} e^{\alpha t}$$

$$\Rightarrow \quad y = e^{\alpha t} \int \frac{t^2}{2} dt = \frac{t^3}{6} e^{\alpha t}$$

$$= t^2 \left(\frac{t}{6}\right) e^{\alpha t} = t^m R_1(t) e^{\alpha t}$$

$\Rightarrow$ If α is a repeated root

then (ODE) has a particular solution

of the form $Y(t) = t^m R_1(t) e^{\alpha t}$.

Conclusion

$$y'' + ay' + by = P_1(t)e^{\alpha t} \quad \text{(ODE)}$$

$$r^2 + ar + b = 0 \quad \text{(CE)}$$

$m \triangleq$ multiplicity of α as a root of (CE)

$$= \begin{cases} 0 & \text{if } \alpha \text{ is not a root} \\ 1 & \text{if } \alpha \text{ is a root, but not repeat} \\ 2 & \text{if } \alpha \text{ is a repeat root} \end{cases}$$

Then (ODE) has a particular solution of the form

$$Y(t) = t^m R_1(t) e^{\alpha t}$$

More generally

$$y'' + ay' + by = P_n(t) e^{\alpha t}$$

has a particular solution of the form

$$Y(t) = t^m R_n(t) e^{\alpha t}$$

m = multiplicity of α as a root of (CE)

Remark

- If α Complex, then $e^{\alpha t}$ complex
  $\Rightarrow Y(t)$ complex-valued.

But, we need real-valued solutions

Final Conclusion (Method of Undetermined Coefficients)

- **Case I**

$$y'' + ay' + by = P_n(t)e^{\alpha t}, \quad \alpha \in \mathbb{R} \qquad \text{(ODE)}$$

$$r^2 + ar + br = 0 \qquad \text{(CE)}$$

m = multiplicity of α as a root of (CE)

$$= \begin{cases} 0 & \text{if } \alpha \text{ is not a root} \\ 1 & \text{if } \alpha \text{ is a root but not repeated} \\ 2 & \text{if } \alpha \text{ is a repeated root} \end{cases}$$

$$Y(t) = t^m Q_n(t) e^{\alpha t}$$

- **Case II**

$$y'' + ay' + by = P_n(t)\cos\beta t\, e^{\alpha t} \text{ or } P_n(t)\sin\beta t\, e^{\alpha t} \quad (\beta \neq 0)$$

($\beta \neq 0$, otherwise reduces to Case I.)

m = multiplicity of $\alpha+i\beta$ as a root of (CE)

$$= \begin{cases} 0 & \text{if } \alpha+i\beta \text{ is not a root} \\ 1 & \text{if } \alpha+i\beta \text{ is a root} \end{cases}$$

$$Y(t) = t^m\left(Q_n(t)\cos\beta t + R_n(t)\sin\beta t\right)e^{\alpha t}.$$

Ex Find a particular solution of

$$y'' + y' - 2y = \begin{cases} (i) & t+1 \\ (ii) & (t+2)e^{-2t} \\ (iii) & -10\sin t \\ (iv) & e^{t}\cos t \end{cases}$$

Sol. The characteristic equation

$$r^2 + r - 2 = 0, \quad r_1 = 1, \quad r_2 = -2.$$

(i) $g(t) = P_1(t)e^{0t}$

0 is not a root $\Rightarrow m = 0$

$$Y(t) = Q_1(t) = At + B$$

Substituting $Y(t)$ into the (ODE) $\Longrightarrow$

$$(At+B)'' + (At+B)' - 2(At+B) = t+1$$

$\Longrightarrow$

$$-2At + A - 2B = t+1$$

$\Longrightarrow$

$$\begin{cases} -2A = 1 \\ A - 2B = 1 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{2} \\ B = -\frac{3}{4} \end{cases}$$

so $Y(t) = -\frac{t}{2} - \frac{3}{4}$

(iii) $g(t) = (t+2)e^{-2t} = P_1(t)e^{-2t}$ $\alpha = -2$

$\alpha = r_2$, $r_2 \neq r_1 \Rightarrow m = 1$

$$Y(t) = tQ_1(t)e^{-2t} = t(At+B)e^{-2t}.$$

Substituting into (ODE) yields

$$e^{-2t}(-6At + 2A - 3B) = (t+2)e^{-2t}$$

$$\Longrightarrow \begin{cases} -6A = 1 \\ 2A - 3B = 2 \end{cases} \Rightarrow \begin{cases} A = -\frac{1}{6} \\ B = -\frac{7}{9} \end{cases}$$

$$\Rightarrow Y(t) = \left(-\frac{t^2}{6} - \frac{7}{9}t\right)e^{-2t}.$$

(iii) $g(t) = -10\sin t = -10 \sin t\, e^{0t}$

$\alpha + i\beta = 0 + i$ is not a root $\Rightarrow m = 0$

$Y(t) = A\cos t + B\sin t$

Substituting $Y(t)$ into ODE yields

$(-3A+B)\cos t + (-A-3B)\sin t = -10\sin t$

$$\Rightarrow \begin{cases} -3A + B = 0 \\ -A - 3B = -10 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 3. \end{cases}$$

$\Rightarrow Y(t) = \cos t + 3\sin t$

(iv) $g(t) = P_0(t)\cos t\, e^{t}$

$\alpha + i\beta = 1 + i$ is not a root of (CE)

$\Rightarrow m = 0$

$Y(t) = (A\cos t + B\sin t)e^{t}$

Substituting Y into ODE yields

$[(8B - A)\cos t - (B + 3A)\sin t]e^{t} = \cos t\, e^{t}$

$$\Rightarrow$$

$$\begin{cases} 3B - A = 1 \\ B + 3A = 0 \end{cases} \rightarrow \begin{cases} A = -\frac{1}{10} \\ B = \frac{3}{10} \end{cases}$$

$$\Rightarrow Y(t) = \left(-\frac{\cos t}{10} + \frac{3}{10}\sin t\right) e^t.$$

__Remark__ If $g(t) = g_1(t) + \cdots + g_k(t)$

$$y'' + ay' + by = g(t)$$

A particular solution $Y(t)$ can be decomposed as

$$Y(t) = Y_1(t) + Y_2(t) + \cdots + Y_k(t),$$

where

$$Y_i''(t) + aY_i' + bY_i = g_i(t), \quad i = 1, 2, \ldots, k.$$

__Ex__ Find a particular solution of

$$y'' + y' - 2y = -10\sin t + e^t \cos t$$

__Sol__ $g_1 = -10\sin t$, $g_2 = e^t \cos t$.

$$Y(t) = Y_1(t) + Y_2(t)$$

For Y_1

$$Y_1'' + Y_1' - 2Y_1 = -10\sin t$$

By [illegible], we can choose

$$Y_1(t) = \cos t + 3\sin t$$

For Y_2

$$Y_2'' + Y_2' - 2Y_2 = e^t \cos t,$$

By [illegible], we get

$$Y_2(t) = \left(-\frac{\cos t}{10} + \frac{3}{10}\sin t\right)e^t$$

$$\Rightarrow Y(t) = Y_1 + Y_2$$

$$= \cos t + 3\sin t + \left(-\frac{\cos t}{10} + \frac{3}{10}\sin t\right)e^t. \#$$

Go.

(a) $y'' - 2y' - 3y = 6e^{2t}$ $\quad (y = -2e^{2t} + C_1e^{3t} + C_2e^{-t})$

(b) $y'' + 9y = t^2e^{3t} + 18$ $\quad (y = C_1\cos 3t + C_2\sin 3t + 2 + (\frac{t^2}{18} - \frac{t}{27} + \frac{1}{162})e^{3t}$

(c) $2y'' + 3y' + y = t^2 + 3\cos t$ $\quad (y = C_1e^{-t} + C_2e^{-\frac{t}{2}} + t^2 - 6t + 14 - \frac{3}{10}\cos t + \frac{9}{10}\sin t$