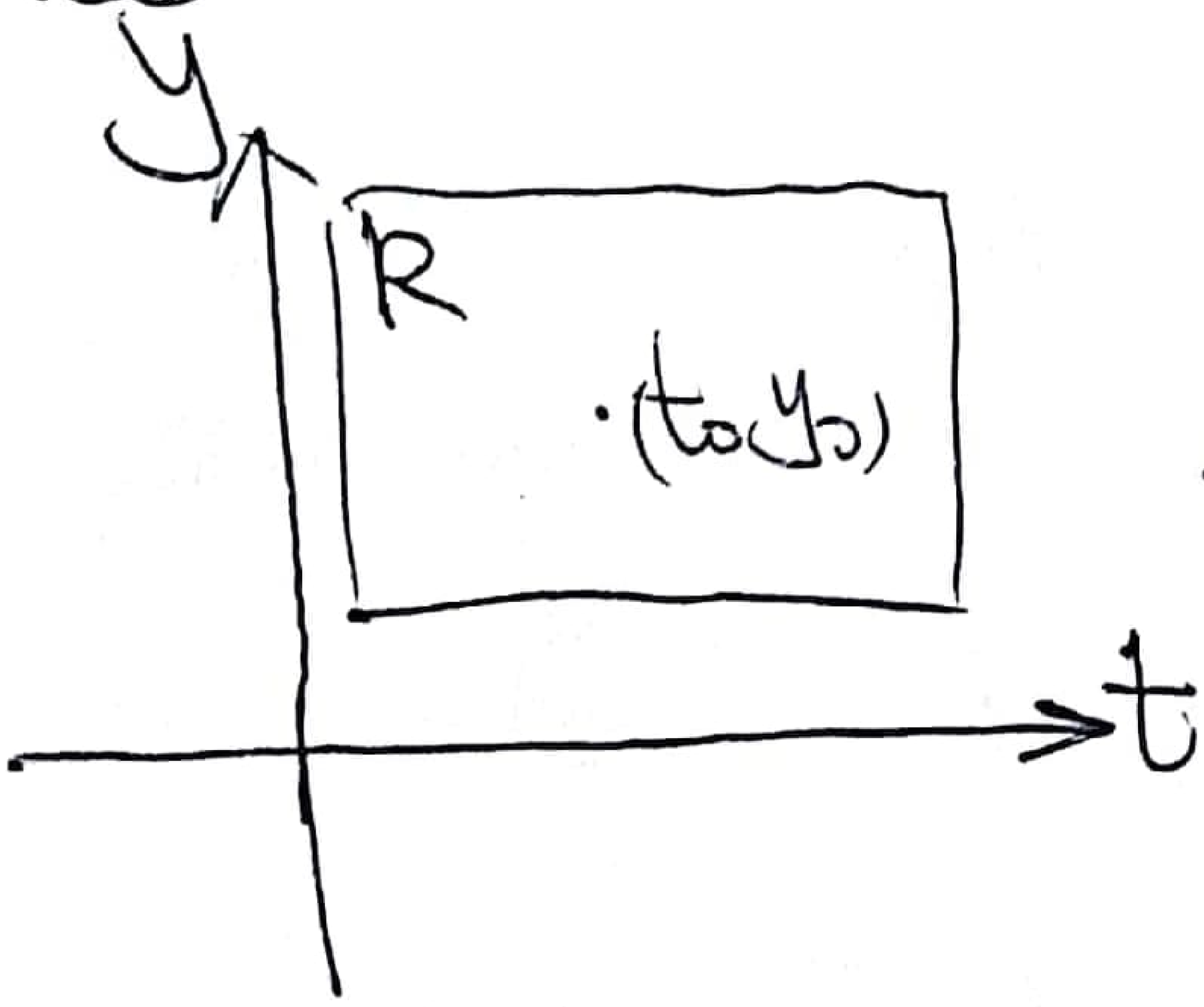
2.8 The existence and uniqueness

Consider the (IVP)

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases} \qquad \text{(IVP)}$$

Recall



$f, \frac{\partial f}{\partial y}$ Continuous on R

$\Longrightarrow$ local $\exists!$

We have more accurate

Theorem 2.8.1 (Existence and Uniqueness) Let R be a closed rectangle

$$R = \{(t, y) \mid |t - t_0| \le a,\ |y - y_0| \le b\},\ a, b > 0.$$

Assume that f is continuous on R and

$$|f(t, x) - f(t, y)| \le L |x - y| \qquad (\ast)$$

for all $t \in [t_0 - a, t_0 + a]$ and $x, y \in [y_0 - b, y_0 + b]$, $L > 0$ is a positive constant. Then (IVP) has a unique solution on the time interval (t_0-h, t_0+h), where $h=\min\{\frac{b}{M}, \frac{1}{2L}\}$, and $M=\max_{(t,y)\in R}|f(t,y)|$.

<u>Idea</u>

- (IVP) is equivalently to (Exercise)

$$y(t)=y_0+\int_{t_0}^{t} f(s,y(s))ds. \qquad (*)$$

- If we get an "<u>Approximate Solution</u>" φ_n to $(*)$, then we can expect

$$\varphi_{n+1}(t) \triangleq y_0+\int_{t_0}^{t} f(s,\varphi_n(s))ds \qquad (**)$$

is a better Approximate Solution.

- Suppose that

$$\lim_{n\to\infty}\varphi_n(t)=\varphi(t). \qquad \text{(I)}$$

Then, <u>formally</u> by $(**)$

$$\lim_{n\to\infty}\varphi_{n+1}(t)=\lim_{n\to\infty}\int_{t_0}^{t} f(s,\varphi_n(s))ds+y_0$$

$$\overset{?}{=} \int_{t_0}^{t} \lim_{n\to\infty} f(s,\varphi_n(s))ds+y_0$$

$$=\int_{t_0}^{t} f(s,\varphi(s))ds+y_0$$

$\Longrightarrow$

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\,ds$$

$$\varphi(t_0) = y_0$$

$\Longrightarrow$ $\exists$ Solution

$\boxed{\text{Warning}}$

① In order that φ_{n+1} given by (**) is well-defined, we need

$$|\varphi_n(t) - y_0| \leq b \qquad (4)$$

② The exchange of order

$$\lim_{n\to\infty} \int_{t_0}^{t} \longrightarrow \int_{t_0}^{t} \lim_{n\to\infty} \ ?$$

Recall the fact:

If $\varphi_n(t) \to \varphi(t)$ uniformly in $t \in [t_0-h, t_0+h]$, (3)

then

$$\lim_{n\to\infty} \int_{t_0}^{t} f(s, \varphi_n(s))\,ds = \int_{t_0}^{t} f(s, \varphi(s))\,ds.$$

In Conclusion

We need (**) & (1) & (2).

Proof. ~~Construct~~

Step 1 Construct (**) "approximate solutions" $\{\varphi_n\}_{n=0}^{\infty}$ satisfying (1).

to be determined.

$$\varphi_0(t) \equiv y_0, \quad t \in (t_0 - h, t_0 + h)$$

Obviously φ_0 satisfies (1). Assume that we already have φ_n satisfying (1), then we conductively define φ_{n+1} as (**), i.e.

$$\varphi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\,ds, \quad t \in (t_0 - h, t_0 + h).$$

Then

$$|\varphi_{n+1}(t) - y_0| = \left| \int_{t_0}^{t} f(s, \varphi_n(s))\,ds \right|$$

$$\leq M|t - t_0| \leq Mh \leq b,$$

i.e. φ_{n+1} satisfies (1). Therefore, we get a sequence of "approximate solutions" $\{\varphi_n\}_{n=0}^{\infty}$ defined on $[t_0 - h, t_0 + h]$, satisfies (**) and (1).

Step 2 Uniform Convergence of $\{\varphi_n\}_{n=0}^{\infty}$.

$$\varphi_{n+1} = (\varphi_{n+1}-\varphi_n)+(\varphi_n-\varphi_{n-1})+\cdots+(\varphi_2-\varphi_1)+(\varphi_1-\varphi_0)+\varphi_0$$

(2.1) ~~E~~stimate on $|\varphi_{k+1}-\varphi_k|$.

$$\begin{aligned}
&|\varphi_{k+1}(t)-\varphi_k(t)| \\
&= \left|\int_{t_0}^{t}\left(f(s,\varphi_k(s))-f(s,\varphi_{k-1}(s))\right)ds\right| \leftarrow \text{By } (**) \\
&\leq \int_{t_0}^{t_0+h}|f(s,\varphi_k(s))-f(s,\varphi_{k-1}(s))|\,ds \\
&\leq L\int_{t_0}^{t_0+h}|\varphi_k(s)-\varphi_{k-1}(s)|\,ds \leftarrow \text{By } (L) \\
&\leq Lh \underbrace{\sup_{t_0-h\leq t\leq t_0+h}|\varphi_k(t)-\varphi_{k-1}(t)|}_{A_k} \\
&\leq \frac{A_k}{2}, \quad t\in[t_0,t_0+h].
\end{aligned}$$

Similarly

$$|\varphi_{k+1}(t)-\varphi_k(t)|\leq\frac{A_k}{2}, \quad t\in[t_0-h,t_0].$$

$\Longrightarrow$

$$A_{k+1}\triangleq\max_{t_0-h\leq t\leq t_0+h}|\varphi_{k+1}(t)-\varphi_k(t)|\leq\frac{A_k}{2}, \quad k=1,2,\cdots.$$

$\Longrightarrow$

$$A_{k+1}\leq\frac{A_k}{2}\leq\frac{A_{k-1}}{2^2}\leq\cdots\leq\frac{A_1}{2^k},$$

$\Rightarrow$

$$|\varphi_{k+1}(t) - \varphi_k(t)| \leq A_{k+1} \leq \frac{A_1}{2^k}, \quad k=1,2,\cdots \quad (3)$$

(2.2) Estimate $|\varphi_n(t) - \varphi_m(t)|$ $\quad (n \geq m+1)$

$$\varphi_n(t) - \varphi_m(t) = (\varphi_n(t) - \varphi_{n-1}(t)) + (\varphi_{n-1}(t) - \varphi_{n-2}(t)) + \cdots + (\varphi_{m+1}(t) - \varphi_m(t))$$

$\Rightarrow$

$$|\varphi_n(t) - \varphi_m(t)| \leq |\varphi_n(t) - \varphi_{n-1}(t)| + |\varphi_{n-1}(t) - \varphi_{n-2}(t)| + \cdots + |\varphi_{m+1}(t) - \varphi_m(t)|$$

$$\leq \frac{A_1}{2^{n-1}} + \frac{A_1}{2^{n-2}} + \cdots + \frac{A_1}{2^m} \quad \leftarrow \text{By (3)}$$

$$\leq \frac{A_1}{2^{m-1}} \quad (4)$$

(2.3) Uniform Convergence

By (4) $\{\varphi_n(t)\}_{n=1}^{\infty}$ is a Cauchy sequence, for $t_0-h \leq t \leq t_0+h$.

$\Rightarrow \exists\, \varphi(t)$, s.t

$$\lim_{n\to\infty} \varphi_n(t) = \varphi(t), \quad t \in [t_0-h, t_0+h]. \quad (5)$$

Let $n\to\infty$ in (4) $\Rightarrow$

$$|\varphi_m(t) - \varphi(t)| \leq \frac{A_1}{2^{m-1}}$$

$\Longrightarrow$

$$\varphi_m(t) \xrightarrow{1} \varphi(t), \quad t \in [t_0-h, t_0+h]. \qquad (6)$$

<u>Step 3</u> Existence

Recall (**), i.e.

$$\varphi_{n+1}(t) = y_0 + \int_{t_0}^{t} f(s, \varphi_n(s))\,ds, \quad t \in [t_0-h, t_0+h]$$

$\Longrightarrow$

$$\lim_{n\to\infty} \varphi_{n+1}(t) = y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(s, \varphi_n(s))\,ds$$

$$= y_0 + \int_{t_0}^{t} \lim_{n\to\infty} f(s, \varphi_n(s))\,ds \Leftarrow \text{By (6)}$$

$$= y_0 + \int_{t_0}^{t} f(s, \varphi(s))\,ds$$

$\Longrightarrow$

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\,ds, \quad \forall t \in [t_0-h, t_0+h].$$

$$\varphi(t_0) = y_0$$

$\Longrightarrow$ Existence.

<u>Step 4</u> Uniqueness

Let $x(t)$ and $y(t)$ be two solutions $\Longrightarrow$

$$x(t) - y(t) = \int_{t_0}^{t} \big(f(s, x(s)) - f(s, y(s))\big)\,ds$$

$\Longrightarrow$

$$|x(t)-y(t)| = \left|\int_{t_0}^{t}\left(f(s,x(s))-f(s,y(s))\right)ds\right|$$

$$\leq \int_{t_0}^{t}|f(s,x(s))-f(s,y(s))|\,ds \qquad (7)$$

$$\leq L\int_{t_0}^{t}|x(s)-y(s)|\,ds \Leftarrow \text{By } (L)$$

Set

$$f(t) = \int_{t_0}^{t}|x(s)-y(s)|\,ds$$

then

$$f'(t) = |x(t)-y(t)|.$$

By (7) $\Longrightarrow$

$$f'(t) \leq Lf(t)$$

$$\Longrightarrow e^{-Lt}f'(t) - Le^{-Lt}f(t) \leq 0$$

i.e.

$$\left(e^{-Lt}f(t)\right)' \leq 0, \qquad t\in[t_0, t_0+h]$$

$\Longrightarrow$

$$e^{-Lt}f(t) \leq e^{-Lt_0}f(t_0) = 0$$

$$\Longrightarrow f(t) \leq 0$$

$$\Longrightarrow |x(t)-y(t)| \leq Lf(t) \leq 0 \Leftarrow \text{By } (7)$$

$$\Longrightarrow x(t) = y(t) \Longrightarrow \text{Uniqueness.}$$ #

8.

Chapter 3 Second Order Linear Equations

2^{nd} order linear ODE

- general form

  $P(t)y'' + Q(t)y' + R(t)y = g(t) \qquad (P(t) \not\equiv 0)$

  Homogeneous: $g \equiv 0$; Nonhomogeneous $g \not\equiv 0$.

  No general approach to find solutions

  Solvable Cases (being learned)

  1. Exact equations
  2. Euler equations

- With Constant Coefficient

  $y'' + ay' + by = g(t)$

  Homogeneous Case: Completely solvable

  Nonhomogeneous Case: $\begin{cases} \text{Undetermined Coefficients} \\ \text{Variation of Parameters} \end{cases}$

3.1 Homogeneous Equations with Constant Coefficients

- Preliminaries

(i) Imaginary unit i

$$i \triangleq \sqrt{-1} \qquad \sqrt{-100} = \sqrt{100 \times (-1)} = \sqrt{100} \times \sqrt{-1}$$
$$= 10i$$

(ii) Euler's formula

$$e^{i\theta} = \cos\theta + i\sin\theta, \quad \theta \in \mathbb{R}$$

$$\Rightarrow e^{\lambda + i\mu} = e^{\lambda} e^{i\mu} = e^{\lambda}(\cos\mu + i\sin\mu), \quad \mu \in \mathbb{R}.$$

Consider

$$y'' + ay' + by = 0. \qquad \text{(ODE)}$$

Solution ?

Try the solution of the form $e^{rt} = y(t)$, then Substituting this for y in (ODE) leads to

$$(r^2 + ar + b)e^{rt} = 0$$

$$\Rightarrow$$

$r^2 + ar + b = 0 \Leftarrow$ Characteristic Equation of (ODE).

$$r^2 + ar + b = 0 \qquad (CE)$$

Three possible cases:

- Case I: $(a^2 - 4b > 0)$

  (CE) has two real roots $r_1 \neq r_2$

  $\Rightarrow$ (ODE) has two solutions $y_1 = e^{r_1 t}$, $y_2 = e^{r_2 t}$.

- Case II $(a^2 - 4b < 0)$

  (CE) has two conjugate complex roots

  $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$, $\lambda \in \mathbb{R}, \mu \in \mathbb{R}, \mu \neq 0$.

  $\Rightarrow$ (ODE) has two (complex-valued) solutions

$$y_1^c = e^{r_1 t} = e^{(\lambda + i\mu)t}$$

$$= e^{\lambda t}\left(\cos(\mu t) + i\sin(\mu t)\right)$$

$$y_2^c = e^{r_2 t} = e^{(\lambda - i\mu)t}$$

$$= e^{\lambda t}\left(\cos(\mu t) - i\sin(\mu t)\right)$$

$\Rightarrow$ (ODE) has two (real-valued) solutions

$$y_1^R = e^{\lambda t}\cos(\mu t), \quad y_2^R = e^{\lambda t}\sin(\mu t).$$

Case III $(a^2-4b=0)$

(CE) has repeat real roots $r_1=r_2=r\in\mathbb{R}$.

$\Rightarrow$ (ODE) has one solution of the form e^{rt}.

Question: For Case III, how to find another solution $y\notin\{\phi(t)\,|\,\phi(t)=ce^{rt},\ c\in\mathbb{R}\}$?

~~Observe~~

Observe: If $\phi(t)$ is a solution to (ODE), then

$C\,\phi(t)$ is also a solution to (ODE).

$\downarrow$ ~~Variation of parameters~~

$C(t)\,\phi(t)$?

Back to Case III:

e^{rt} is a solution

$y(t)\overset{?}{=}C(t)\,e^{rt}$ another solution?

$$
\begin{aligned}
0 &= y'' + ay' + by \\
&= (c(t)e^{rt})'' + a(c(t)e^{rt})' + b\,c(t)e^{rt} \\
&= c''(t)e^{rt} + 2r\,c'(t)e^{rt} + r^2 c(t)e^{rt} \\
&\quad + a\,c'(t)e^{rt} + ar\,c(t)e^{rt} + b\,c(t)e^{rt} \\
&= e^{rt}\left(c''(t) + 2r\,c'(t) + a\,c'(t)\right) \\
&\quad + e^{rt}(r^2 + ar + b)\,c(t) \leftarrow (\text{CE}) \\
&= e^{rt}\left(c''(t) + (2r+a)\,c'(t)\right)
\end{aligned}
$$

$$\Rightarrow \boxed{c''(t) + (2r+a)\,c'(t) = 0}$$

1st order linear ODE in $c'(t)$!

$$\Rightarrow c'(t) = e^{-(2r+a)t}$$

$$\Rightarrow c(t) = \int e^{-(2r+a)t}\,dt.$$

Recall, for Case II $a^2 = 4b$

$\Rightarrow$ (CE): $r^2 + ar + \frac{a^2}{4} = 0$

$\Rightarrow r = -\frac{a}{2}$

$\Rightarrow$ $C(t) = \int e^{-(-a+a)t} dt = t$

$\Rightarrow$ $\boxed{y(t) = te^{rt}}$

Therefore,

<u>Case II</u> $a^2 - 4b = 0$

$\Rightarrow$ (ODE) has two solutions

$y_1 = e^{rt}$

$y_2 = te^{rt}$.

Conclusion

$y'' + ay' + by = 0$ (ODE)

$r^2 + ar + b = 0$ (CE)

Case I ($a^2 - 4b > 0$) $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$

$y_1 = e^{r_1 t}$ $y_2 = e^{r_2 t}$

$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, $c_1, c_2 \in \mathbb{R}$;

Case II ($a^2 - 4b < 0$) $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$

$y_1 = e^{\lambda t}\cos(\mu t)$ $y_2 = e^{\lambda t}\sin(\mu t)$

$y = e^{\lambda t}(c_1\cos(\mu t) + c_2\sin(\mu t))$, $c_1, c_2 \in \mathbb{R}$

Case III ($a^2 - 4b = 0$) $r_1 = r_2 = r \in \mathbb{R}$

$y_1 = e^{rt}$ $y_2 = te^{rt}$

$y = e^{rt}(c_1 + c_2 t)$, $c_1, c_2 \in \mathbb{R}$.

Ex 1 Find the general solution of

(a) $y''+5y'+6y=0$

(b) $y''-2y'+8y=0$

(c) $y''+4y'+4y=0$

Sol. (a) The characteristic equation is

$$r^2+5r+6=0$$

$$\Rightarrow (r+2)(r+3)=0,$$

$$r_1=-2,\ r_2=-3$$

$$\Rightarrow y_1=e^{-2t},\quad y_2=e^{-3t}$$

$\Rightarrow$ general solution is $y=c_1e^{-2t}+c_2e^{-3t},\ c_1,c_2\in\mathbb{R}$

(b) The characteristic equation is

$$r^2-2r+8=0$$

The roots

$$r_1=\frac{2+\sqrt{4-32}}{2}=1+\sqrt{7}i$$

$$r_2=1-\sqrt{7}i$$

Therefore, we have two solutions

$$y_1 = e^t \cos(\sqrt{7}t), \quad y_2 = e^t \sin(\sqrt{7}t),$$

and general solution

$$y = e^t\left(C_1\cos(\sqrt{7}t) + C_2\sin(\sqrt{7}t)\right), \quad C_1, C_2 \in \mathbb{R}.$$

(3) The characteristic equation is

$$r^2 + 4r + 4 = 0$$

We have repeat root $r = -2$. Then, one has two solutions

$$y_1 = e^{-2t} \quad y_2 = te^{-2t}$$

and general solution

$$y = e^{-2t}(C_1 + C_2 t), \quad C_1, C_2 \in \mathbb{R}.$$

3.5 Nonhomogeneous Equations: Method of Undetermined Coefficients

$$y'' + ay' + by = g(t) \qquad \text{(NE)}$$

nonhomogeneous term.

If $Y(t)$ is a particular solution to (NE). Then for any solution y to (NE) we have

$$\phi(t) = y(t) - Y(t)$$

satisfies

$$\phi'' + a\phi' + b\phi = 0 \qquad \text{(HE)}$$

Observe that (HE) can be solved completely as before. as

$$\phi(t) = \underbrace{C_1 y_1(t) + C_2 y_2(t)}_{\text{general solution to (HE)}}, \quad C_1, C_2 \in \mathbb{R}$$

$\Longrightarrow$

a particular solution to (NE)

$$y(t) = \underbrace{Y(t) + C_1 y_1(t) + C_2 y_2(t)}_{\text{general solution to (NE)}}, \quad C_1, C_2 \in \mathbb{R}.$$

gen

Set the function space X as the sum, subtraction, or product of exponential functions, polynomial functions, sines, or cosines. Then

$$\frac{d^k f}{dt^k} \in X, \text{ for all } f \in X$$

$\Longrightarrow$

$$f'' + af' + bf \in X, \text{ if } f \in X.$$

Or

$$y'' + ay' + by = g(t) \in X, \text{ if } y \in X.$$

Therefore, if

| $g(t)$ is the sum, subtraction, or product of exponential, polynomials, sines, or cosines |

then one may find a particular solution $Y(t)$ in the same space, s.t

$$y'' + ay' + by = g(t).$$

Ex1 Find a particular solution of

$$y''-3y'-4y=3e^{2t}$$

Sol. Expect solution of the form $y=Ae^{2t}$. Then

$$y''-3y'-4y=A\left(2^2-3\times 2-4\right)e^{2t}$$

$$=-6Ae^{2t}=3e^{2t}$$

We get $A=-\frac{1}{2}$, and thus find a particular solution

$$Y(t)=-\frac{1}{2}e^{2t}.$$

Ex1 Find a particular solution of

$$y''-3y'-4y=2\sin t$$

Sol. ① ~~Expect~~ Try solution of form $Y(t)=A\sin t$

then

$$Y''-3Y'-4Y=A\left(-\sin t-3\cos t-4\sin t\right)$$

$$=A\left(-3\cos t-5\sin t\right)\neq 2\sin t$$

No such A!!

② Try Solution of the form

$Y(t) = A\sin t + B\cos t$

$Y'' - 3Y' - 4Y$

$= A(-\sin t - 3\cos t - 4\sin t)$

$+ B(-\cos t + 3\sin t - 4\cos t)$

$= (3B - 5A)\sin t - (3A + 5B)\cos t$

$= 2\sin t$

We need

$$\begin{cases} 3B - 5A = 2 \\ 3A + 5B = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{-5}{17} \\ B = \frac{3}{17} \end{cases}$$

$\Rightarrow$

$Y(t) = -\frac{5}{17}\sin t + \frac{3}{17}\cos t.$

Ex Find a particular solution of

$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$

Sol Try solution

$$Y(t) = (A\cos 2t + B\sin 2t)e^{t}$$

$\Rightarrow$ $Y'(t) = (A+2B)e^{t}\cos 2t + (B-2A)e^{t}\sin 2t$

$Y''(t) = (4B-3A)e^{t}\cos 2t - (4A+3B)e^{t}\sin 2t$

$\Rightarrow$ Substituting Y'' and Y' into the ODE $\Rightarrow$

$$\begin{cases} 10A + 2B = 8 \\ 2A - 10B = 0 \end{cases}$$

$\Rightarrow$ $A = \frac{10}{13}$ $B = \frac{2}{13}$

$\Rightarrow$ $Y(t) = \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$

Ex Find a particular solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

Sol. $Y = Y_1 + Y_2 + Y_3$

$$Y_1'' - 3Y_1' - 4Y_1 = 3e^{2t}$$

$$Y_2'' - 3Y_2' - 4Y_2 = 2\sin t$$

$$Y_3'' - 3Y_3' - 4Y_3 = -8e^{t}\cos 2t$$

$$Y_1 = -\frac{1}{2}e^{2t}$$

$$Y_2 = -\frac{5}{17}\sin t + \frac{3}{17}\cos t$$

$$Y_3 = \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$

$\Rightarrow Y = \cdots$

Ex Find a particular solution to

$$y'' - 3y' - 4y = 2e^{-t}$$

Sol. Expect $Y(t) = Ae^{-t}$. Then

$$Y'' - 3Y' - 4Y = A(1+3-4) = 0$$

$$\neq 2e^{-t} \;!$$

$\Rightarrow$ No solution of the form Ae^{-t} !

$\Rightarrow$ Need other kind of solution !

Check

(CE): $r^2 - 3r - 4 = 0$

$r = -1$ is a root of (CE).