MATH 3270B - Ordinary Differential Equations - 2017/18

Quiz 2

Time allowed: 45 mins

NAME: ______________________________ ID: ______________________________

Answer all the questions. Show your detailed steps.

1. Let $W(t)$ be the Wronskian of two solutions $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ of

$$\tfrac{d}{dt}\mathbf{x} = \begin{pmatrix} 1 & \alpha \\ -\alpha & 1+2\alpha \end{pmatrix} \mathbf{x},$$

where α is a constant.

(a) Assume that $W(t)$ satisfies

$$W'(t) = 4W(t).$$

Find the α.

(b) Let α be obtained in (a), find the fundamental matrix $\Phi(t)$ satisfying $\Phi(0) = I$ of the linear system;

(c) Find the unique solution of the linear system with initial value

$$\mathbf{x}(0) = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{8} \end{pmatrix}.$$

Solution:

(a)

$$4 = TrP(t) = 1 + 1 + 2\alpha \implies \alpha = 1.$$

(b) Due to (a), we have

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix},$$

and thus solving the characteristic equation

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 4 = 0$$

gives

$$\lambda_1 = \lambda_2 = 2.$$

Direct calculations lead to

$$A - \lambda_1 I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (A - \lambda_1 I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Solving $(A - \lambda_1 I)^2 \vec{r_0} = 0$:

$$\vec{r_0}^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{r_0}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and

$$\vec{r_1}^{(1)} = (A - \lambda_1 I)\vec{r_0}^{(1)} = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$
$$\vec{r_1}^{(2)} = (A - \lambda_1 I)\vec{r_0}^{(2)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then

$$\vec{x}^{(1)} = (\vec{r_0}^{(1)} + t\vec{r_1}^{(1)})e^{\lambda_1 t} = \begin{pmatrix} 1-t \\ -t \end{pmatrix} e^{2t}$$
$$\vec{x}^{(2)} = (\vec{r_0}^{(2)} + t\vec{r_1}^{(2)})e^{\lambda_2 t} = \begin{pmatrix} t \\ t+1 \end{pmatrix} e^{2t}$$

are two solutions. The Wronskian

$$W(\vec{x}^{(1)}, \vec{x}^{(2)})(t) = \begin{vmatrix} 1-t & t \\ -t & t+1 \end{vmatrix} e^{4t} = e^{4t} \neq 0.$$

Therefore $\{\vec{x}^{(1)}, \vec{x}^{(2)}\}$ is a fundamental set of solutions.

$$\Psi(t) = (\vec{x}^{(1)}(t), \vec{x}^{(2)}(t)) = \begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix} e^{2t}$$

is a fundamental matrix. Noticing that $\Psi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\Phi(t) := \Psi(t)$ is a fundamental matrix with $\Phi(0) = I$.

(c) The unique solution is

$$\vec{X} = \Phi(t)\vec{X}(0) = e^{2t} \begin{pmatrix} 1-t & t \\ -t & t+1 \end{pmatrix} \begin{pmatrix} \frac{1}{4} \\ \frac{1}{8} \end{pmatrix} = e^{2t} \begin{pmatrix} \frac{1}{4} - \frac{t}{8} \\ -\frac{t}{8} + \frac{1}{8} \end{pmatrix}.$$

2. Consider the system:

$$\frac{d}{dt}\mathbf{x} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 2 & 1 \\ -2 & 1 & 3 \end{pmatrix} \mathbf{x}.$$

(a) Find a fundamental matrix $\Psi(t)$ for the above homogeneous system;

(b) Find the unique solution $\mathbf{x}(t)$ of the above system, with initial condition

$$\mathbf{x}(0) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution:

(a) Solving the characteristic equation

$$det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 2-\lambda & 1 \\ -2 & 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(\lambda-2)(\lambda-3) = 0,$$

one obtains $\lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$

For $\lambda_1 = 1$,

$$(A - \lambda_1 I)\vec{r_1} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ -2 & 1 & 2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

obtaining an eigenvector

$$\vec{r_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

and the corresponding solution

$$\vec{x}^{(1)} = \vec{r_1} e^{\lambda_1 t} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{t}.$$

For $\lambda_2 = 2$,

$$(A - \lambda_2 I)\vec{r_2} = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

obtaining an eigenvector

$$\vec{r_2} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

and the corresponding solution

$$\vec{x}^{(2)} = \vec{r_2} e^{\lambda_2 t} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t}.$$

For $\lambda_3 = 3$,

$$(A - \lambda_3 I)\vec{r_3} = \begin{pmatrix} -2 & 1 & 0 \\ -1 & -1 & 1 \\ -2 & 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

obtaining an eigenvector

$$\vec{r_3} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

and the corresponding solution

$$\vec{x}^{(3)} = \vec{r_3} e^{\lambda_3 t} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} e^{3t}.$$

The Wronskian

$$W(t) = W(\vec{x}^{(1)}, \vec{x}^{(2)}, \vec{x}^{(3)})(t) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix} e^{6t} = 2e^{6t} \neq 0.$$

Therefore $\Psi(t) = \begin{pmatrix} e^t & e^{2t} & e^{3t} \\ 0 & e^{2t} & 2e^{3t} \\ e^t & e^{2t} & 3e^{3t} \end{pmatrix}$ is a fundamental matrix.

(b)

$$\Psi(0) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \quad and, \; \Psi^{-1}(0) = \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 1 & 1 & -1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

Therefore

$$\vec{X}(t) = \Psi(t)\Psi^{-1}(0)\vec{X}(0) = \begin{pmatrix} e^{3t} - e^{2t} - e^t \\ 2e^{3t} - e^{2t} \\ 3e^{3t} - e^{2t} - e^t \end{pmatrix}.$$

3. Consider the system:

$$\frac{d}{dt}\mathbf{x} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix} \mathbf{x}.$$

(a) Find a fundamental matrix $\Psi(t)$ for the above homogeneous system;

(b) Find the unique solution $\mathbf{x}(t)$ of the above system, with initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Solution:

(a) Solving the characteristic equation

$$det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)[(1-\lambda)(3-\lambda)+1] = -(\lambda-2)^3 = 0,$$

one obtains $\lambda_1 = \lambda_2 = \lambda_3 = 2$. Direct calculations yield

$$A - \lambda_1 I = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ (A-\lambda_1 I)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$(A-\lambda_1 I)^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Solving $(A-\lambda_1 I)^3 \vec{r_0} = 0$ yields

$$\vec{r_0}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{r_0}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{r_0}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and correspondingly

$$\vec{r_1}^{(1)} = (A-\lambda_1 I)\vec{r_0}^{(1)} = \begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}, \quad \vec{r_1}^{(2)} = (A-\lambda_1 I)\vec{r_0}^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix},$$

$$\vec{r_1}^{(3)} = (A-\lambda_1 I)\vec{r_0}^{(3)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

and

$$\vec{r_2}^{(1)} = (A-\lambda_1 I)\vec{r_1}^{(1)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{r_2}^{(2)} = (A-\lambda_1 I)\vec{r_1}^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\vec{r_2}^{(3)} = (A-\lambda_1 I)\vec{r_1}^{(3)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

We have three solutions

$$\vec{x}^{(1)} = (\vec{r_0}^{(1)} + t\vec{r_1}^{(1)} + \frac{t^2}{2}\vec{r_2}^{(1)})e^{2t} = \begin{pmatrix} 1-t \\ -t \\ 0 \end{pmatrix} e^{2t},$$

$$\vec{x}^{(2)} = \begin{pmatrix} t \\ t+1 \\ 0 \end{pmatrix} e^{2t}, \quad \vec{x}^{(3)} = \begin{pmatrix} \frac{t^2}{2} \\ \frac{t^2}{2}+t \\ 1 \end{pmatrix} e^{2t}.$$

The Wronskian

$$W(t) = \begin{vmatrix} 1-t & t & \frac{t^2}{2} \\ -t & t+1 & \frac{t^2}{2}+t \\ 0 & 0 & 1 \end{vmatrix} e^{6t} = e^{6t} \neq 0.$$

Therefore

$$\Psi(t) = \begin{pmatrix} 1-t & t & \frac{t^2}{2} \\ -t & t+1 & \frac{t^2}{2}+t \\ 0 & 0 & 1 \end{pmatrix} e^{2t}$$

is a fundamental matrix.

(b)

$$\Psi(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then

$$\vec{X}(t) = \Psi(t)\vec{X}(0) = e^{2t} \begin{pmatrix} 1-t & t & \frac{t^2}{2} \\ -t & t+1 & \frac{t^2}{2}+t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e^{2t} \begin{pmatrix} \frac{t^2}{2}+1 \\ \frac{t^2}{2}+t+1 \\ 1 \end{pmatrix}$$

4. (a) Solve the initial value problem

$$\begin{cases} y' + ty = ty^2, \\ y(0) = y_0. \end{cases}$$

(b) Find the largest interval J, such that the unique solution to (a) exits on $\mathbb{R}$, for any $y_0 \in J$.

Answer:

(a) If $y_0 = 0$, then $y = 0$ is a solution on $\mathbb{R}$. If $y_0 \neq 0$, $y \neq 0$, divide the equation by y^2,

$$\frac{y'}{y^2} + \frac{t}{y} = t,$$

let $z = \frac{1}{y}$,

$$\begin{aligned}
-z' + tz =& t \\
(e^{-\frac{1}{2}t^2} z)' =& -te^{-\frac{1}{2}t^2} \\
e^{-\frac{1}{2}t^2} z =& -\int_0^t se^{-\frac{1}{2}s^2} ds + z_0 \\
=& e^{-\frac{1}{2}t^2} - 1 + \frac{1}{y_0} \\
\frac{1}{y} =& 1 - e^{\frac{1}{2}t^2} + \frac{e^{\frac{1}{2}t^2}}{y_0} \\
y =& \frac{1}{1 + (\frac{1}{y_0} - 1)e^{\frac{1}{2}t^2}}
\end{aligned}$$

(b) If $y_0 \neq 0$,

$$\begin{aligned}
1 + (\frac{1}{y_0} - 1)e^{\frac{1}{2}t^2} \neq& 0 \\
\frac{1}{y_0} \neq& 1 - e^{-\frac{1}{2}t^2} \\
\frac{1}{y_0} \notin& [0, 1) \\
y_0 \in& (-\infty, 0) \cup (0, 1]
\end{aligned}$$

Thus the largest interval $J = (-\infty, 1]$.