THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Solution of Tutorial Classwork 0

1. (a) By definition, we have $\emptyset, X \in \mathfrak{T}$.

   Pick arbitrary set of elements $\{U_\alpha\}_{\alpha\in I}$ of $\mathfrak{T}$. By definition, $X\backslash U_\alpha$ is countable for any $\alpha \in I$. Note that $X\backslash(\bigcup_{\alpha\in I} U_\alpha) = \bigcap_{\alpha\in I}(X\backslash U_\alpha)$. Since subset of countable set is also countable and $X\backslash(\bigcup_{\alpha\in I} U_\alpha) \subset X\backslash U_{\alpha_0}$ for some $\alpha_0 \in I$, the set $X\backslash(\bigcup_{\alpha\in I} U_\alpha)$ is countable. Hence $\bigcup_{\alpha\in I} U_\alpha \in \mathfrak{T}$.

   Pick finitely many elements $\{U_i\}_{i=1}^n$ of $\mathfrak{T}$. By definition, $X\backslash U_i$ is countable for any $i = 1, 2, \dots n$. Note that $X\backslash(\bigcap_{i=1}^n U_i) = \bigcup_{i=1}^n (X\backslash U_i)$. Since finite union of countable set is also countable, the set $X\backslash(\bigcap_{i=1}^n U_i)$ is countable. Hence $\bigcap_{i=1}^n U_i \in \mathfrak{T}$.

   As a result, $\mathfrak{T}$ is a topology.

   (b) To show that $\mathfrak{T}$ is not a metric topology, it suffices to show that $\mathfrak{T}$ is not Hausdorff. Pick any two distinct elements $x, y \in X$. Suppose there exists two open sets U, V such that $x \in U, y \in V$ and $U \cap V = \emptyset$. By definition, $X\backslash U$ and $X\backslash V$ are countable. This implies that $(X\backslash U) \cup (X\backslash V) = X\backslash(U \cap V) = X$ is also countable, contradicting to the fact that X is uncountable. Therefore X is not Hausdorff and it is not a metric topology.

   (c) If X is countable, then every complement of subsets of X are countable. Hence $\mathfrak{T}$ is the discrete topology. The discrete topology is induced by the discrete metric: $d(x, x) = 0$, $d(x, y) = 1$ for any $x, y \in X, x \neq y$.

2. To show that $\mathfrak{T}_d \subset \mathfrak{T}_\rho$, we need to show that for any $U \in \mathfrak{T}_d$, we have $U \in \mathfrak{T}_\rho$. Recall that for metric space, $U \in \mathfrak{T}_\rho$ is open if and only if for any $x \in U$, we can find a ball $B_\rho(x, \delta)$ for some $\delta > 0$ such that $x \in B_\rho(x, \delta) \subset U$.

   To find such a ball, pick any $x \in U$. Since $U \in \mathfrak{T}_d$, we can find a ball $B_d(x, \epsilon)$ with $\epsilon > 0$ such that $x \in B_d(x, \epsilon) \subset U$.

   Sketch of idea: We want to put a ρ-metric ball $B_\rho(x, \delta)$ inside the d-metric ball $B_d(x, \epsilon)$. In other word, we need to show that for any $y \in B_\rho(x, \delta)$, we have $y \in B_d(x, \epsilon)$. To show that $y \in B_d(x, \epsilon)$, let's consider $d(x, y)$. Since $d(x, y) \leq k\rho(x, y) < k\delta$ for any $y \in B_\rho(x, \delta)$, if we take $\delta = \epsilon/k$, we have $d(x, y) < \epsilon$. Try to write it down mathematically.

3. (a) It is clear that $\rho(x,y)=0 \iff x=y$ and $\rho(x,y)=\rho(y,x)$. The triangle inequality follows easily from the triangle inequality of absolute value:

$$\begin{aligned}\rho(x,y) &= |\tan x - \tan y| \\ &= |(\tan x - \tan z) + (\tan z - \tan y)| \\ &\leq |(\tan x - \tan z)| + |(\tan z - \tan y)| \\ &= \rho(x,z) + \rho(z,y)\end{aligned}$$

(b) Since $f'(x) = \sec^2 x - 1 \geq 0$ for any $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the function $f(x)$ is increasing.

To show that $\mathfrak{T}_d \subset \mathfrak{T}_\rho$, pick any $x, y \in (-\frac{\pi}{2}, \frac{\pi}{2})$. WLOG assume that $x < y$. By the previous result, we know that $f(x) \leq f(y)$. This implies $y - x \leq \tan y - \tan x$. Hence $d(x,y) \leq \rho(x,y)$. By Exercise 2, we have $\mathfrak{T}_d \subset \mathfrak{T}_\rho$.

To show that $\mathfrak{T}_\rho \subset \mathfrak{T}_d$, we want to put a d-metric ball $B_d(x,\delta)$ inside the ρ-metric ball $B_\rho(x,\epsilon)$. More precisely, given any $\epsilon > 0$. We need to find $\delta > 0$ such that whenever $y \in B_d(x,\delta)$, i.e. $d(x,y) = |x-y| < \delta$, we have $\rho(x,y) = |\tan x - \tan y| < \epsilon$. Why does this property hold?

(c) (X,d) is incomplete since the Cauchy sequence $\left\{(\frac{\pi}{2} - \frac{1}{n})\right\}_{n\in\mathbb{N}}$ does not converge in the metric d.

For the metric space (X,ρ), given any Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ in ρ, the sequence $\{\tan x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the standard metric. By completeness of $\mathbb{R}$, we know that the sequence $\tan x_n \to L \in \mathbb{R}$ as $n \to \infty$. Hence we have $x_n \to \tan^{-1} L \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and (X,ρ) is complete.

(d) *Sketch of idea:* From (c), the reason why the open subset $(-\frac{\pi}{2}, \frac{\pi}{2})$ is incomplete is that the boundary point is removed. So whenever a Cauchy sequence approaching to the boundary, it must diverge. To fix this problem, we would like to define a metric which enlarges the distance around the boundary points (compare $\left|(\frac{\pi}{2} - \frac{1}{n}) - (\frac{\pi}{2} - \frac{1}{m})\right|$ and $\left|\tan\left(\frac{\pi}{2} - \frac{1}{n}\right) - \tan\left(\frac{\pi}{2} - \frac{1}{m}\right)\right|$ for large m, n). In particular, if we can measure the distance from a point $x \in A$ and the boundary, then the reciprocal of this distance is what we want.

The "distance from a point x to the boundary" can be defined by

$$d(x, X\backslash A) = \inf_{z \in X\backslash A} d(x,z)$$

One clever definition of the desired metric is given by

$$\tau(x,y) = d(x,y) + \left| \frac{1}{d(x, X\backslash A)} - \frac{1}{d(y, X\backslash A)} \right|$$

Try to show that this metric satisfies the required property.