THE CHINESE UNIVERSITY OF HONG KONG

DEPARTMENT OF MATHEMATICS

MATH3070 (Second Term, 2017–2018)

Introduction to Topology

Exercise 12 Fundamental group

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $\alpha, \beta, \gamma\colon\ I = [0,1] \to X$ be paths with $\alpha(1) = \beta(0)$ and $\beta(1) = \gamma(0)$. Show that $H\colon\ I \times [0,1] \to X$ is a homotopy rel $\{0,1\}$ for $(\alpha * \beta) * \gamma \simeq \alpha * (\beta * \gamma)$ where

$$H(s,t) = \begin{cases} \alpha\left(\frac{s}{a(t)}\right) & s \in [0, a(t)] \\ \beta\left(\frac{s-a(t)}{b(t)-a(t)}\right) & s \in [a(t), b(t)] \\ \gamma\left(\frac{s-b(t)}{1-b(t)}\right) & s \in [b(t), 1]\ , \end{cases}$$

   for any continuous functions $1/4 \le a(t) < b(t) \le 3/4$ with $a(0) = 1/4$, $a(1) = 1/2$, $b(0) = 1/2$, $b(1) = 3/4$.

   Furthermore, give an explicit example of $a(t)$ and $b(t)$.

2. Let α be a loop in X based at $x_0 \in X$ and c be the constant path at x_0. Give explicit homotopies to show that

$$\alpha * c \simeq \alpha \simeq c * \alpha \quad \text{rel} \quad \{0,1\}.$$

   Do you have a similar statement when α is not a loop?

3. Prove that inverse exists in $\pi_1(X, x_0)$ with product $*$.

4. Let X be a path connected space and $x_0, x_1 \in X$. Find an isomorphism between the groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$.

5. Show that if X and Y are of the same homotopy type, then $\pi_1(X) = \pi_1(Y)$.

6. Show that the mapping $\varphi:\ \mathbb{R}^2 \setminus \{\,\mathbf{0}\,\} \to \mathbb{S}^1$ where $\varphi(\mathbf{x}) = \mathbf{x}/\left\|\mathbf{x}\right\|$ is a retraction. Modify φ to show that $\mathbb{S}^1$ is indeed a deformation retract.

7. Is your method above applicable to $\mathbb{S}^n \subset \mathbb{R}^{n+1} \setminus \{\,\mathbf{0}\,\}$?

8. Use winding number argument to show that $\pi_1(\mathbb{S}^1) = \pi_1(\mathbb{C} \setminus \{0\}) = (\mathbb{Z}, +)$.

9. Give the outline of an argument that $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$.

10. Formulate the major steps in proving every loop in the torus is a product of the two generating loops, α and β.

11. Formulate the major steps in proving every loop in the projective plane is a product of the generating loops.

12. Convince yourself intuitively that the fundamental group of the Klein Bottle is $(\mathbb{Z} \oplus \mathbb{Z}/2, +)$.

13. Let $\mathbb{S}^1 \vee \mathbb{S}^1$ be the space of two circles with one point glued to each other. Convince yourself that $\mathbb{S}^1 \vee \mathbb{S}^1$ is a deformation retract of a punctured torus.

    Remark. This is the simplest example of spaces with non-abelian fundamental group. Their fundamental group is a free group on two generators.

14. Let X be a topological space and $\alpha_1, \ldots, \alpha_n$ are loops based at $x_0 \in X$. Show that $[\alpha_1] \cdots [\alpha_n] = 1$ in $\pi_1(X, x_0)$ if and only if the loop $\alpha_1 * \cdots * \alpha_n$ is homotopic to a loop γ at x_0 such that γ is the boundary of a disk in X.

15. Apply the above result to the case of the torus, $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$, show that

$$\pi_1(\mathbb{T}) = \langle a, b : \ aba^{-1}b^{-1} = 1 \rangle = (\mathbb{Z} \oplus \mathbb{Z}, +) .$$

16. Further, show that if Σ_g is the compact orientable surface of genus g, then

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g : \ a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle.$$