THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2017–2018)
Introduction to Topology
Exercise 5 Convergence

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let (x_n) be a sequence in (X, d) such that $d(x_n, x) \to c \in \mathbb{R}$ for a point $x \in X$. Can we conclude the convergence of (x_n)?

2. Given a sequence (x_n) and A be the set of points $\{x_n\}$.
   (a) Give an example of (x_n) that it converges and $\overline{A} \neq A$.
   (b) If $\overline{A} = A$, can you conclude anything about the convergence of (x_n)? Justify your conclusion by proof or examples.

3. Formulate a statement about the convergence of a sequence in $X \times Y$ (with product topology) with reference to the convergence of sequences in X and Y.

4. Let (X, d) be a metric space and two sequences in X satisfy $x_n \to x$ and $y_n \to y$. Show that $d(x_n, y_n) \to d(x, y)$.

5. Let (X, d) be a metric space. Show that if a sequence $x_n \to x$ then every subsequence of it converges to x. Show also the converse that if every convergent subsequence of (x_n) converges to x then $x_n \to x$. Is it true for general topological spaces.

6. Let X be a first countable space. Show that $x \in \overline{A}$ if and only if there is a sequence (a_n) in A converging to x. Moreover, show that $f\colon X \to Y$ is continous at $x_0 \in X$ if and only if for all sequence (x_n) converging to x_0, the sequence $f(x_n)$ converges to $f(x_0)$.

7. Let $\mathbb{R}_{\ell\ell}$, $\mathbb{R}_{cf}$ and $\mathbb{R}$ be the real line with lower limit topology, cofinite topology, and standard topology respectively. Find examples of sequences that converge in one topology but not in another.

8. By placing the lower limit topology, cofinite topology, or standard topology at suitable place, could you find examples of mappings $f : \mathbb{R} \to \mathbb{R}$ such that every sequence $x_n \to x$ satisfies $f(x_n) \to f(x)$ but the function is not continuous at x.