According to definition (2) of $f(z)$,

$$\left|\int_{C_\rho} f(z)\,dz\right| \le \frac{\rho^{-a}}{1-\rho}2\pi\rho = \frac{2\pi}{1-\rho}\rho^{1-a}$$

and

$$\left|\int_{C_R} f(z)\,dz\right| \le \frac{R^{-a}}{R-1}2\pi R = \frac{2\pi R}{R-1}\cdot\frac{1}{R^a}.$$

Since $0 < a < 1$, the values of these two integrals evidently tend to 0 as ρ and R tend to 0 and ∞, respectively. Hence, if we let ρ tend to 0 and then R tend to ∞ in equation (4), we arrive at the result

$$(1 - e^{-i2a\pi})\int_0^\infty \frac{r^{-a}}{r+1}\,dr = 2\pi i e^{-ia\pi},$$

or

$$\int_0^\infty \frac{r^{-a}}{r+1}\,dr = 2\pi i\frac{e^{-ia\pi}}{1-e^{-i2a\pi}}\cdot\frac{e^{ia\pi}}{e^{ia\pi}} = \pi\frac{2i}{e^{ia\pi}-e^{-ia\pi}}.$$

Using the variable of integration x here, instead of r, as well as the expression

$$\sin a\pi = \frac{e^{ia\pi}-e^{-ia\pi}}{2i},$$

we arrive at the desired result:

$$(5)\qquad \int_0^\infty \frac{x^{-a}}{x+1}\,dx = \frac{\pi}{\sin a\pi} \qquad (0 < a < 1).$$

EXERCISES

1. Use the function $f(z) = (e^{iaz} - e^{ibz})/z^2$ and the indented contour in Fig. 108 (Sec. 89) to derive the integration formula

$$\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2}\,dx = \frac{\pi}{2}(b-a) \qquad (a \ge 0, b \ge 0).$$

Then, with the aid of the trigonometric identity $1 - \cos(2x) = 2\sin^2 x$, point out how it follows that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}.$$

2. Derive the integration formula

$$\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{\sqrt{2}}$$

by integrating the function

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

over the indented contour appearing in Fig. 109 (Sec. 90).

3. Derive the integration formula obtained in Exercise 2 by integrating the branch

$$f(z) = \frac{z^{-1/2}}{z^2+1} = \frac{e^{(-1/2)\log z}}{z^2+1} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

of the multiple-valued function $z^{-1/2}/(z^2+1)$ over the closed contour in Fig. 110 (Sec. 91).

4. Derive the integration formula

$$\int_0^\infty \frac{\sqrt[3]{x}}{(x+a)(x+b)}\,dx = \frac{2\pi}{\sqrt{3}} \cdot \frac{\sqrt[3]{a}-\sqrt[3]{b}}{a-b} \qquad (a > b > 0)$$

using the function

$$f(z) = \frac{z^{1/3}}{(z+a)(z+b)} = \frac{e^{(1/3)\log z}}{(z+a)(z+b)} \qquad (|z| > 0, 0 < \arg z < 2\pi)$$

and a closed contour similar to the one in Fig. 110 (Sec. 91), but where

$$\rho < b < a < R.$$

5. The ***beta function*** is this function of two real variables:

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt \qquad (p > 0, q > 0).$$

Make the substitution $t = 1/(x+1)$ and use the result obtained in the example in Sec. 91 to show that

$$B(p, 1-p) = \frac{\pi}{\sin(p\pi)} \qquad (0 < p < 1).$$

6. Consider the two simple closed contours shown in Fig. 111 and obtained by dividing into two pieces the annulus formed by the circles C_ρ and C_R in Fig. 110 (Sec. 91). The legs L and $-L$ of those contours are directed line segments along any ray $\arg z = \theta_0$, where $\pi < \theta_0 < 3\pi/2$. Also, Γ_ρ and γ_ρ are the indicated portions of C_ρ, while Γ_R and γ_R make up C_R.

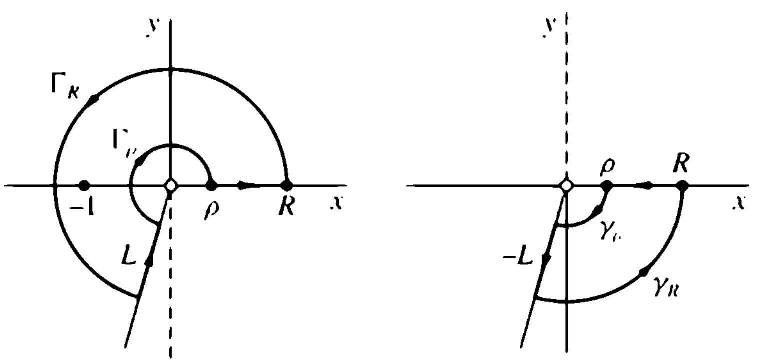

FIGURE 111

(*a*) Show how it follows from Cauchy's residue theorem that when the branch

$$f_1(z) = \frac{z^{-a}}{z+1} \qquad \left(|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2}\right)$$

at a and 0, respectively. The singularity $z = 1/a$ is, of course, exterior to the circle C since $|a| < 1$.

Inasmuch as

$$f(z) = \frac{\phi(z)}{z-a} \quad \text{where} \quad \phi(z) = \frac{z^4+1}{(az-1)z^2},$$

it is easy to see that

$$B_1 = \phi(a) = \frac{a^4+1}{(a^2-1)a^2}. \tag{10}$$

The residue B_2 can be found by writing

$$f(z) = \frac{\phi(z)}{z^2} \quad \text{where} \quad \phi(z) = \frac{z^4+1}{(z-a)(az-1)};$$

and straightforward differentiation reveals that

$$B_2 = \phi'(0) = \frac{a^2+1}{a^2}. \tag{11}$$

Finally, by substituting the residues (10) and (11) into expression (9), we arrive at the integration formula (8).

EXERCISES

Use residues to establish the following integration formulas:

1. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} = \frac{2\pi}{3}.$

2. $\displaystyle\int_{-\pi}^{\pi} \frac{d\theta}{1+\sin^2\theta} = \sqrt{2}\pi.$

3. $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta\, d\theta}{5-4\cos 2\theta} = \frac{3\pi}{8}.$

4. $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}} \quad (-1 < a < 1).$

5. $\displaystyle\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{a\pi}{\left(\sqrt{a^2-1}\right)^3} \quad (a > 1).$

6. $\displaystyle\int_0^{\pi} \sin^{2n}\theta\, d\theta = \frac{(2n)!}{2^{2n}(n!)^2}\pi \quad (n = 1, 2, \ldots).$

93. ARGUMENT PRINCIPLE

A function f is said to be ***meromorphic*** in a domain D if it is analytic throughout D except for poles. Suppose now that f is meromorphic in the domain interior to a positively oriented simple closed contour C and that it is analytic and nonzero on C.

EXERCISES

1. Let C denote the unit circle $|z| = 1$, described in the positive sense. Use the theorem in Sec. 93 to determine the value of $\Delta_C \arg f(z)$ when

(*a*) $f(z) = z^2$; (*b*) $f(z) = 1/z^2$; (*c*) $f(z) = (2z - 1)^7/z^3$.

Ans. (*a*) 4π; (*b*) -4π; (*c*) 8π.

2. Let f be a function which is analytic inside and on a positively oriented simple closed contour C, and suppose that $f(z)$ is never zero on C. Let the image of C under the transformation $w = f(z)$ be the closed contour Γ shown in Fig. 114. Determine the value of $\Delta_C \arg f(z)$ from that figure; and, with the aid of the theorem in Sec. 93, determine the number of zeros, counting multiplicities, of f interior to C.

Ans. 6π; 3.

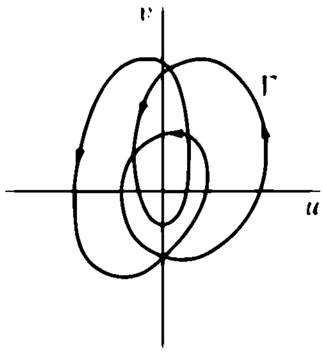

FIGURE 114

3. Using the notation in Sec. 93, suppose that Γ does not enclose the origin $w = 0$ and that there is a ray from that point which does not intersect Γ. By observing that the absolute value of $\Delta_C \arg f(z)$ must be less than 2π when a point z makes one cycle around C and recalling that $\Delta_C \arg f(z)$ is an integral multiple of 2π, point out why the winding number of Γ with respect to the origin $w = 0$ must be zero.

4. Suppose that a function f is meromorphic in the domain D interior to a simple closed contour C on which f is analytic and nonzero, and let D_0 denote the domain consisting of all points in D except for poles. Point out how it follows from the lemma in Sec. 28 and Exercise 11, Sec. 83, that if $f(z)$ is not identically equal to zero in D_0, then the zeros of f in D are all of finite order and that they are finite in number.

Suggestion: Note that if a point z_0 in D is a zero of f that is not of finite order, then there must be a neighborhood of z_0 throughout which $f(z)$ is identically equal to zero.

5. Suppose that a function f is analytic inside and on a positively oriented simple closed contour C and that it has no zeros on C. Show that if f has n zeros z_k $(k = 1, 2, \ldots, n)$ inside C, where each z_k is of multiplicity m_k, then

$$\int_C \frac{zf'(z)}{f(z)}\,dz = 2\pi i \sum_{k=1}^{n} m_k z_k.$$

[Compare with equation (8), Sec. 93, when $P = 0$ there.]

6. Determine the number of zeros, counting multiplicities, of the polynomial

(*a*) $z^6 - 5z^4 + z^3 - 2z$; (*b*) $2z^4 - 2z^3 + 2z^2 - 2z + 9$; (*c*) $z^7 - 4z^3 + z - 1$,

inside the circle $|z| = 1$.

Ans. (*a*) 4; (*b*) 0; (*c*) 3.

7. Determine the number of zeros, counting multiplicities, of the polynomial

(*a*) $z^4 - 2z^3 + 9z^2 + z - 1$; (*b*) $z^5 + 3z^3 + z^2 + 1$

inside the circle $|z| = 2$.

Ans. (*a*) 2; (*b*) 5.

8. Determine the number of roots, counting multiplicities, of the equation

$$2z^5 - 6z^2 + z + 1 = 0$$

in the annulus $1 \le |z| < 2$.

Ans. 3.

9. Show that if c is a complex number such that $|c| > e$, then the equation $cz^n = e^z$ has n roots, counting multiplicities, inside the circle $|z| = 1$.

10. Let two functions f and g be as in the statement of Rouché's theorem in Sec. 94, and let the orientation of the contour C there be positive. Then define the function

$$\Phi(t) = \frac{1}{2\pi i}\int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\,dz \qquad (0 \le t \le 1)$$

and follow these steps below to give another proof of Rouché's theorem.

(*a*) Point out why the denominator in the integrand of the integral defining $\Phi(t)$ is never zero on C. This ensures the existence of the integral.

(*b*) Let t and t_0 be any two points in the interval $0 \le t \le 1$ and show that

$$|\Phi(t) - \Phi(t_0)| = \frac{|t - t_0|}{2\pi}\left|\int_C \frac{fg' - f'g}{(f + tg)(f + t_0 g)}\,dz\right|.$$

Then, after pointing out why

$$\left|\frac{fg' - f'g}{(f + tg)(f + t_0 g)}\right| \le \frac{|fg' - f'g|}{(|f| - |g|)^2}$$

at points on C, show that there is a positive constant A, which is independent of t and t_0, such that

$$|\Phi(t) - \Phi(t_0)| \le A|t - t_0|.$$

Conclude from this inequality that $\Phi(t)$ is continuous on the interval $0 \le t \le 1$.

(*c*) By referring to equation (8), Sec. 93, state why the value of the function Φ is, for each value of t in the interval $0 \le t \le 1$, an integer representing the number of zeros of $f(z) + tg(z)$ inside C. Then conclude from the fact that Φ is continuous, as shown in part (*b*), that $f(z)$ and $f(z) + g(z)$ have the same number of zeros, counting multiplicities, inside C.

95. INVERSE LAPLACE TRANSFORMS

Suppose that a function F of the complex variable s is analytic throughout the finite s plane except for a finite number of isolated singularities. Then let L_R denote a vertical line segment from $s = \gamma - iR$ to $s = \gamma + iR$, where the constant γ is positive and large enough that the singularities of F all lie to the left of that segment (Fig. 115). A