eg1 $f(z) = \frac{\sinh z}{1+z}$, $|z|<1$

(Find 1st few terms of the Taylor's series of f up to z^4.)

Soln: $f(z) = (\sinh z)\left(\frac{1}{1+z}\right)$

$$= \left(z + \frac{z^3}{3!} + \cdots\right)\left(1 - z + z^2 - z^3 + \cdots\right)$$
$(|z|<1)$

$$= z - z^2 + z^3 - z^4 + \cdots$$
$$+ \frac{z^3}{3!} - \frac{z^4}{3!} + \cdots$$

$$= z - z^2 + \frac{7}{6}z^3 - \frac{7}{6}z^4 + \cdots \quad (|z|<1)$$

#

eg2 Division

$$\frac{1}{\sinh z} = \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots} = \frac{1}{z} \cdot \frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}$$

$$= \frac{1}{z} \cdot \frac{1}{1 + \frac{z^2}{3!}\left(1 + \frac{3!}{5!}z^2 + \cdots\right)}$$

$$= \frac{1}{z} \cdot \left\{1 - \frac{z^2}{3!}\left(1 + \frac{3!}{5!}z^2 + \cdots\right) + \left[\frac{z^2}{3!}\left(1 + \frac{3!}{5!}z^2 + \cdots\right)\right]^2 - \cdots\right\}$$

$$= \frac{1}{z}\left\{ 1 - \frac{z^2}{3!} - \frac{z^4}{5!} - \cdots + \frac{z^4}{(3!)^2}\left(1 + \frac{3!}{5!} z^2 + \cdots\right)^2 - \cdots \right\}$$

$$= \frac{1}{z}\left\{ 1 - \frac{z^2}{3!} - \frac{z^4}{5!} + \frac{z^4}{(3!)^2} + \cdots \right\}$$

$$= \frac{1}{z} - \frac{z}{6} + \frac{7}{120} z^3 + \left(z^5 \text{ or about} \ldots\right)$$

$0 < |z| <$ small enough.

Or

<u>Long division</u>

$$\begin{array}{r|l}
 & 1 - \frac{z^2}{3!} + \frac{7}{120} z^4 - \cdots \\
\hline
1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots & 1 \\
 & 1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots \\
\cline{2-2}
 & -\frac{z^2}{3!} - \frac{z^4}{5!} - \cdots \\
 & -\frac{z^2}{3!} - \frac{z^4}{(3!)^2} - \cdots \\
\cline{2-2}
 & \frac{7}{120} z^4 + \cdots \\
 & \frac{7}{120} z^4 + \cdots \\
\cline{2-2}
 & \cdots
\end{array}$$

$$\therefore \quad \frac{1}{\sinh z} = \frac{1}{z} \cdot \frac{1}{1+\frac{z^2}{3!}+\frac{z^4}{5!}+\cdots}$$

$$= \frac{1}{z}\left(1-\frac{z^2}{3!}+\frac{7}{120}z^4-\cdots\right)$$

$$= \frac{1}{z}-\frac{z}{6}+\frac{7}{120}z^3-\cdots \quad (0<|z|<\pi)$$

Ch6 Residues and Poles

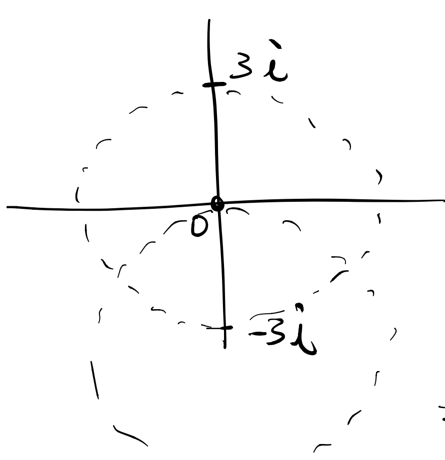
§74 Isolated Singular Points

Def: A singular point z_0 is said to be isolated if there is a deleted ε-neighborhood

$0<|z-z_0|<\varepsilon$ of z_0 throughout which f is analytic

(i.e. no other singular point of f in $0<|z-z_0|<\varepsilon$)

eg1: $f(z)=\dfrac{z-1}{z^5(z^2+9)}$ has isolated singular points at $z=0$, $z=\pm 3i$



f is analytic in

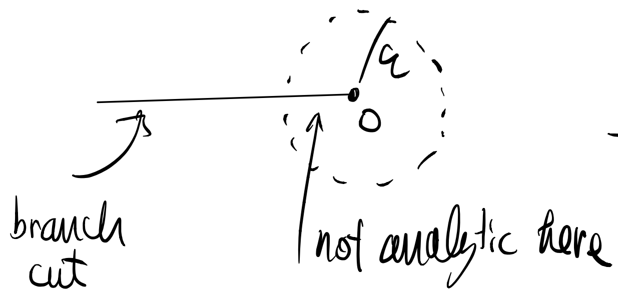
$0<|z|<3$

$0<|z-(-3i)|<3$

& $0<|z-3i|<3$

$\Rightarrow z=0, \pm 3i$ are isolated singularities.

eg2: $\operatorname{Log} z$ (principal branch)



$\Rightarrow$ 0 is not an isolated singular point of $\operatorname{Log} z$.

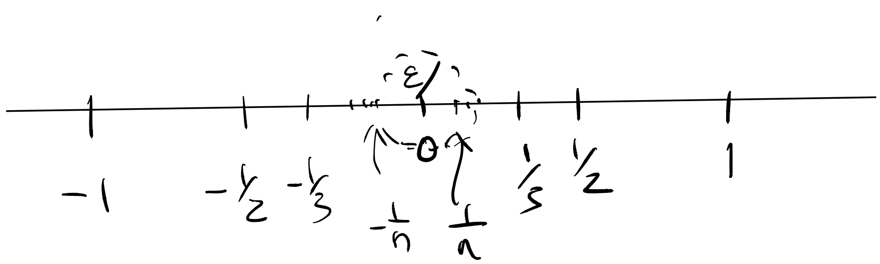
eg3: $f(z) = \dfrac{1}{\sin\left(\frac{\pi}{z}\right)}$

z is a singularity of $f(z)$

$\Leftrightarrow \sin\left(\frac{\pi}{z}\right) = 0$ or $z = 0$ — f is not defined.

$\Leftrightarrow \frac{\pi}{z} = n\pi, \ n = \pm 1, \pm 2 \cdots$ or $z = 0$

$\Leftrightarrow z = \frac{1}{n}, \ n = \pm 1, \pm 3, \cdots$ or $z = 0$



As $z = \frac{1}{n} \to 0$ as $n \to \pm\infty$, $z=0$ is <u>not an isolated</u> singular point of f.

Note, other points $z = \frac{1}{n}$, $n = \pm1, \pm2, \cdots$ are all isolated. (Ex!)

<u>Terminology</u>: If f is analytic in $R_1 < |z| < \infty$, then f is said to have an <u>isolated singular point at $z_0 = \infty$</u>.

§75 <u>Residue</u>

<u>Note</u>: If z_0 is an isolated singular point of f, then f has a Laurent series representation about $z = z_0$

$$f(z) = \cdots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$0 < |z-z_0| < \varepsilon.$$

Def: The **residue** of f at an isolated singular point z_0 is the coefficient $b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$ of the term $\frac{1}{z-z_0}$ in the Laurent expansion of f about z_0, and is denoted by

$$\boxed{\operatorname*{Res}_{z=z_0} f(z) = b_1}$$

(where C = any positively oriented simple closed contour surrounding z_0 & interior to $|z-z_0| = \varepsilon$.)

eg1: Let $f(z) = \dfrac{e^z - 1}{z^5}$.

Then $z=0$ is an isolated singular point of f & f is analytic in $0 < |z| < \infty$.

Laurent expansion

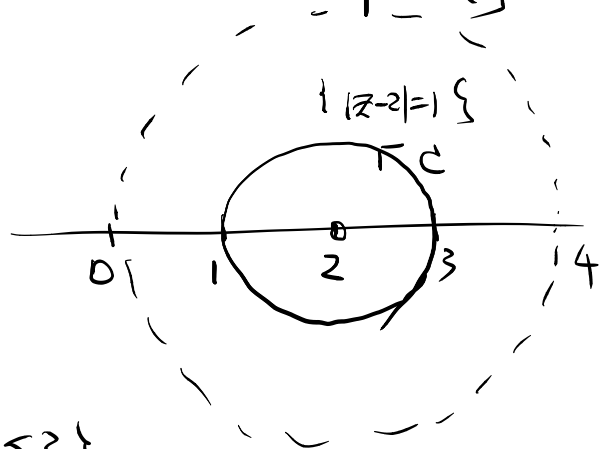
$$f(z) = \frac{1}{z^5}\left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right) - 1\right]$$

$$= \frac{1}{z^4} + \frac{1}{2}\frac{1}{z^3} + \frac{1}{6}\cdot\frac{1}{z^2} + \frac{1}{24}\cdot\frac{1}{z} + \cdots$$

$\Rightarrow \quad \operatorname*{Res}_{z=0} f(z) = \frac{1}{24}$.

Hence $\quad \int_C f(z)\,dz = 2\pi i \frac{1}{24} = \frac{\pi i}{12}$ ※

<u>eg3</u>: Evaluate $\int_C \frac{dz}{z(z-2)^5}$ for $C = \{|z-2| = 1\}$ positively oriented



<u>Soln</u>: $\frac{1}{z(z-2)^5}$

is analytic

in $0 < |z-2| < 2$

Since $C \subset \{0 < |z-2| < 2\}$

and surrounding $z_0 = 2$

$$\therefore \int_C \frac{dz}{z(z-2)^5} = 2\pi i \operatorname{Res}_{z=2} \frac{1}{z(z-2)^5}$$

Consider

$$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5} \cdot \frac{1}{2+(z-2)}$$

$$= \frac{1}{2(z-2)^5} \cdot \frac{1}{1+\left(\frac{z-2}{2}\right)}$$

$$= \frac{1}{2(z-2)^5}\left[1 - \left(\frac{z-2}{2}\right) + \left(\frac{z-2}{2}\right)^2 - \left(\frac{z-2}{2}\right)^3 + \left(\frac{z-2}{2}\right)^4 - \cdots\right]$$

$$= \cdots + \frac{1}{2^5} \cdot \frac{1}{(z-2)} + \cdots$$

$\therefore \ \operatorname{Res}_{z=2} \dfrac{1}{z(z-2)^5} = \dfrac{1}{2^5} = \dfrac{1}{32}$

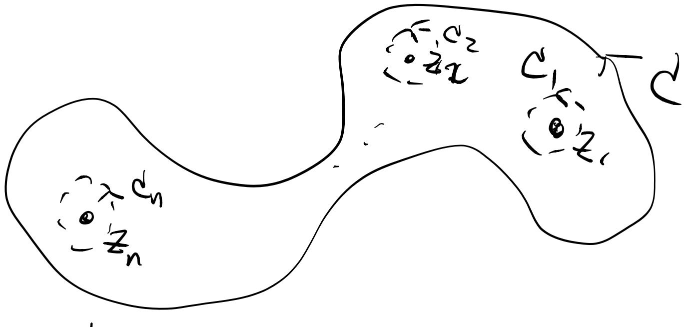
$\therefore \ \displaystyle\int_C \frac{dz}{z(z-2)^5} = 2\pi i \, \frac{1}{32} = \frac{\pi i}{16}$ #

§76 Cauchy's Residue Theorem

Thm: Let C be a positively oriented simple closed contour. If f is analytic inside and on C except finitely many singular points $z_1, \cdots, z_n$ inside C, then

$$\boxed{\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}_{z=z_k} f(z)}$$

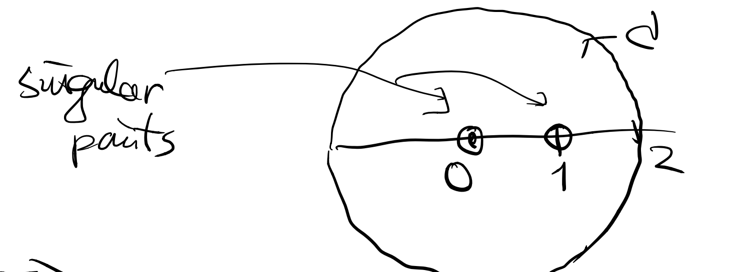
Pf:



Cauchy theorem $\Rightarrow$

$$\int_C f(z)\,dz = \sum_{k=1}^{n} \int_{C_n} f(z)\,dz = \sum_{k=1}^{n} 2\pi i \operatorname{Res}_{z=z_k} f(z)$$ #

eg: Evaluate $\int_C \frac{4z-5}{z(z-1)}\,dz$ for $C = \{|z|=2\}$ positively oriented



Cauchy Residue Thm $\Rightarrow$

$$\int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i \left(\operatorname{Res}_{z=0} \frac{4z-5}{z(z-1)} + \operatorname{Res}_{z=1} \frac{4z-5}{z(z-1)} \right)$$

Ex: $\begin{cases} \operatorname{Res}_{z=0} \frac{4z-5}{z(z-1)} = 5 \\ \operatorname{Res}_{z=1} \frac{4z-5}{z(z-1)} = -1 \end{cases}$ (Note: $\frac{4z-5}{z(z-1)} = \frac{5}{z} + \frac{(-1)}{z-1}$)

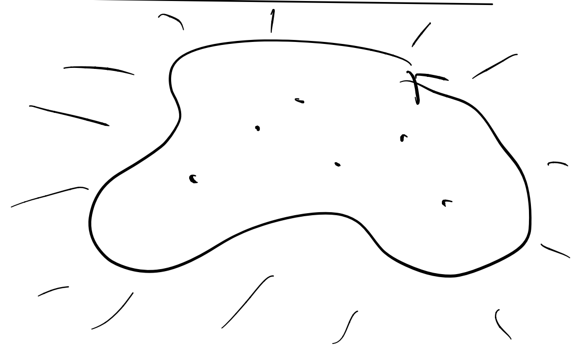
$$\therefore \int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i\,(5-1) = 8\pi i.$$

※

§77 Residue at Infinity

Thm: If f is analytic everywhere in the plane except for a finitely many singular points interior to a positively oriented simple closed contour, then $$\boxed{\int_C f(z)\,dz = 2\pi i \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]}$$

(Pf: Omitted)



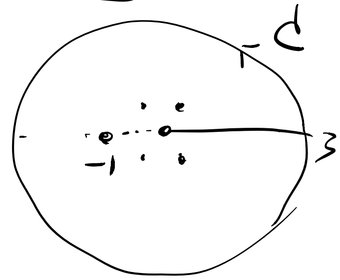
<u>Note</u>:

$$\operatorname{Res}_{z=\infty} f(z) \overset{\text{def}}{=} -\operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right].$$

<u>eg</u>: Evaluate $\displaystyle\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz$

for $C = \{|z|=3\}$ positively oriented

<u>Soln</u>: $f(z) = \dfrac{z^3(1-3z)}{(1+z)(1+2z^4)}$



is analytic except

$z=-1$, $z = \left(-\frac{1}{2}\right)^{1/4}$ with

$|z|=1$, $|z| = \frac{1}{\sqrt[4]{2}} < 3$

$$\Rightarrow \int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz = 2\pi i \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \frac{1}{z^2} \frac{\left(\frac{1}{z}\right)^3\left(1-3\left(\frac{1}{z}\right)\right)}{\left(1+\frac{1}{z}\right)\left(1+2\left(\frac{1}{z}\right)^4\right)}$$

$$= \frac{1}{z^2} \cdot \frac{1}{z^3} \cdot \frac{z-3}{z} \cdot \frac{z}{z+1} \cdot \frac{z^4}{z^4+2}$$

$$= \frac{z-3}{z(1+z)(z^4+2)}$$

$$= \frac{(-3)\left(1-\frac{z}{3}\right)}{2\,z(1+z)\left(1+\frac{z^4}{2}\right)}$$

$$= -\frac{3}{2}\frac{1}{z}\left(1-\frac{z}{3}\right)\left(1-z+z^2-\cdots\right)\left(1-\frac{z^4}{2}+\cdots\right)$$

$$= -\frac{3}{2} \cdot \frac{1}{z} + \cdots$$

$$\therefore \quad \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right] = -\frac{3}{2}$$

Hence $\displaystyle\int_C \frac{z^3(1-3z)}{(1+z)(1+2z^4)}\,dz = 2\pi i\left(-\frac{3}{2}\right) = -3\pi i$

#

§78 The Three Types of Isolated Singular Points

If f is analytic in $0<|z-z_0|<R_2$. Then f has a Laurent series representation about z_0:

$$f(z) = \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

Def: (1) The portion

$$\frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \cdots$$

is called the principal part of f at z_0

(2) Removable singular points

If $b_1 = b_2 = \cdots = b_n = \cdots = 0$, then z_0 is called a removable singular point of f.

(3) Essential Singular Points

If there are infinitely many nonzero b_n in the principal part, then z_0 is called an essential singular point of f.

(4) Poles of order m

If $\exists m \geq 1$ such that $b_m \neq 0$, but

$$b_{m+1} = b_{m+2} = \cdots = 0$$

(i.e. the principal part has finitely many terms)

then z_0 is called a pole of order m of f.

(i.e principal part $= \dfrac{b_1}{z-z_0} + \cdots + \dfrac{b_m}{(z-z_0)^m}$)

A pole of order $m=1$ is called a simple pole.

Notes: (1) z_0 = removable singular point of f

then

$$f(z) = \begin{cases} a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + \cdots, & z \neq z_0 \\ a_0, & z = z_0 \end{cases}$$

defines an analytic function on $|z-z_0| < R_2$

(2) z_0 = pole of order m of f

then

$$f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{z-z_0} + a_0 + a_1(z-z_0) + \cdots$$

$$(0 < |z-z_0| < R_2)$$

$\Rightarrow$

$$(z-z_0)^m f(z) = b_m + \cdots + b_1 (z-z_0)^{m-1} + a_0 (z-z_0)^m + a_1 (z-z_0)^{m+1} + \cdots$$

$= \phi(z)$ is an analytic function on $\{ |z-z_0| < R_2 \}$

with $\phi(z_0) = b_m \neq 0$.

i.e. $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ on $\{ 0 < |z-z_0| < R_2 \}$

with $\phi(z)$ analytic on $\{ |z-z_0| < R_2 \}$

and $\phi(z_0) \neq 0$.

eg1: $f(z) = \dfrac{1-\cosh z}{z^2}$ (analytic in $0 < |z| < \infty$)

$$= \frac{1}{z^2}\left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots \right) \right]$$

$$= \frac{1}{z^2}\left[-\frac{z^2}{2!} - \frac{z^4}{4!} - \cdots \right]$$

$$= -\frac{1}{2} - \frac{z^2}{4!} - \cdots$$

No principal part, $\therefore$ $z=0$ is a removable singular point of $\dfrac{1-\cosh z}{z^2}$

In fact
$$f(z)=\begin{cases} \dfrac{1-\cosh z}{z^2}, & z\neq 0 \\ -\dfrac{1}{2}, & z=0 \end{cases}$$
is an entire function.

eg2: $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots$

infinitely many negative powers of z.

$\Rightarrow$ $z=0$ is an essential singular point of $e^{\frac{1}{z}}$.

#