MATH2040C Linear Algebra II
2017-18 Solution to Homework 4

Exercise 5.A

2 For any $v \in \text{null } S$, we have that $Sx = 0$, then $TSx = 0$, combined with $ST = TS$, we have $STx = 0$, which implies $Tx \in \text{null } S$.

8 If there exist $v = (z, w) \neq (0, 0)$ and $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$, then we have

$$(z, w) = (\lambda w, \lambda z) \Rightarrow z = \lambda w, w = \lambda z \Rightarrow z = \lambda^2 z, w = \lambda^2 w.$$

Since $(z, w) \neq (0, 0)$, we have $\lambda^2 = 1$, then $\lambda = \pm 1$.
For the eigenvalue $\lambda = 1$, $Tv = v$ implies $v \in \text{span}\{(1, 1)\}$; For the eigenvalue $\lambda = -1$, $Tv = -v$ implies $v \in \text{span}\{(1, -1)\}$

15 (a) If λ is an eigenvalue of T, then $\exists$ a non-zero vector $v \in V$, such that $Tv = \lambda v$. Since S is invertible, $\exists$ a non-zero $u \in V$ such that $v = Su$. Then we get $TSu = \lambda Su \Rightarrow S^{-1}TSu = \lambda u \Rightarrow \lambda$ is an eigenvalue of $S^{-1}TS$.
On the other hand, if λ is an eigenvalue of $S^{-1}TS$, then $\exists$ a non-zero vector $v \in V$, such that $S^{-1}TSv = \lambda v \Rightarrow TSv = \lambda Sv$. Let $u = Sv$, then $u \neq 0$ by S being invertible, and $Tu = \lambda u$, which means λ is an eigenvalue of T.

(b) From (a), for any eigenvalue λ of T, if $v \in E(\lambda, T)$(the eigenspace of T with respect to λ, recall Def. 5.36), then

$$Tv = \lambda v \iff S^{-1}TS(S^{-1}v) = \lambda S^{-1}v \iff S^{-1}v \in E(\lambda, S^{-1})$$

Thus we have that v is an eigenvector of T if and only if $S^{-1}v$ is an eigenvector of $S^{-1}TS$.

18* If T has an eigenvalue λ with an eigenvector $v = (z_1, z_2, \ldots)$, then $Tv = \lambda v$ implies that $0 = \lambda z_1, z_1 = \lambda z_2, \ldots, z_n = \lambda z_{n+1}, \ldots$
If $\lambda = 0$, then $z_n = 0$, $\forall n \geq 1$, which contradicts with $v \neq 0$.
And if $\lambda \neq 0$, then we can get $z_1 = 0$, and then $z_2 = 0, z_3 = 0, \ldots$ one by one in turn, which is also a contradiction.

20 Assume that $v = (z_1, z_2, \ldots)$ satisfies $Tv = \lambda v$ for some $\lambda \in \mathbf{F}$, then we have that $z_{n+1} = \lambda z_n,\ \forall\ n \geq 1$, then $z_n = \lambda^{n-1} z_1,\ \forall\ n \geq 1 \Rightarrow v = z_1(1, \lambda, \lambda^2, \ldots)$.
Thus any $\lambda \in \mathbf{F}$ is an eigenvalue of T, and its eigenspace $E(\lambda, T)$ =span$\{(1, \lambda, \lambda^2, \ldots)\}$.

21 (a). If $\lambda \in \mathbf{F}$ with $\lambda \neq 0$, then $\exists$ a non-zero vector v such that $Tv = \lambda v$, then it is equivalent to $T^{-1}v = \lambda^{-1}v$ by T being invertible and $\lambda \neq 0$, which means λ^{-1} is an eigenvalue of T^{-1}.

(b). By (a), we can easily see that for any vector $v \in V$,

$$v \in E(\lambda, T) \iff v \in E(\lambda^{-1}, T^{-1}).$$

29* If T has $k + 2$ distinct eigenvalues, then T has at least $k + 1$ non-zero distinct eigenvalues, denoted by $\lambda_0, \lambda_1, \ldots, \lambda_k$. And assume their eigenvectors as $v_0, v_1, \ldots, v_k$, then by Theorem

5.10, $v_0, v_1, \ldots, v_k$ are linearly independent.
For any vector

$$v = \sum_{i=0}^{k} a_i v_i \in \text{span}(\{v_0, v_1, \ldots, v_k\}),$$

we have that

$$T(\sum_{i=0}^{k} a_i \lambda_i^{-1} v_i) = \sum_{i=0}^{k} a_i \lambda_i^{-1} T v_i = \sum_{i=0}^{k} a_i \lambda_i^{-1} \lambda_i v_i = v.$$

So $\text{span}(\{v_0, v_1, \ldots, v_k\})$ is a $k+1$ dimensional subspace of Range T, which is a contradiction.

32* If we assume $Tf = \lambda f$, then $f' = \lambda f \Rightarrow (e^{-\lambda x} f(x))' = 0 \Rightarrow f(x) = Ce^{\lambda x}$, here C is any constant in $\mathbb{R}$. So any real number $\lambda \in \mathbb{R}$ is an eigenvalue of T with $e^{\lambda x}$ being its eigenvector. Hence by Theorem 5.10, $e^{\lambda_1 x}, \ldots, e^{\lambda_n x}$ is linearly independent.

Exercise 5.B

1* (a). Recall that an operator S is invertible if and only if there exists an operator R such that $SR = I$(or $RS = I$). And note that

$$\begin{aligned}(I-T)(I+T+\cdots+T^{n-1}) &= I+T+\cdots+T^{n-1}-(T+T^2+\cdots+T^n)\\ &= I - T^n = I\end{aligned}$$

(b). Recall that if $x \in \mathbb{R}$, then $1 - x^n = (1-x)(1+x+x^2+\cdots+x^{n-1})$.

3 Since $I - T^2 = 0$, then $(I+T)(I-T) = 0$. Then for any $v \in V$, $(I+V)(I-V)v = 0$. Since -1 is not an eigenvalue of V, then $(I-V)v = 0$, that is $v = Tv$.

9* If λ is a zero of p, then there exists a polynomial $q(x)$, such that $p(x) = (x-\lambda)q(x)$. Then we have $q(T)v \neq 0$, otherwise p is not the polynomial of the smallest degree. And combined with $0 = p(T)v = (T-\lambda)q(T)v$, we have that λ is an eigenvalue of T.

10 $Tv = \lambda v \Rightarrow T^2 v = \lambda^2 v \Rightarrow \ldots \Rightarrow T^n v = \lambda^n v, \ \forall\ n \geq 1$.
So if $p(x) = \sum_{k=0}^{n} a_k x^k$, then we have $p(T)v = \sum_{k=0}^{n} a_k T^k v = \sum_{k=0}^{n} a_k \lambda^k v = p(\lambda)v$.

11 $(\Rightarrow)$: If α is an eigenvalue of $p(T)$, then $\exists\ v \neq 0, s.t.\ p(T)v = \alpha v \Rightarrow (p(T) - \alpha I)v = 0$.
Since we can write $p(x) - \alpha = (x-\lambda_1)\cdots(x-\lambda_n)$ with $\lambda_i \in \mathbb{C}$, then we have

$$(T-\lambda_1)\cdots(T-\lambda_n)v = 0$$

Then if $(T-\lambda_2)\cdots(T-\lambda_n)v \neq 0$, then λ_1 is an eigenvalue of T; Otherwise,
if $(T-\lambda_2)\cdots(T-\lambda_n)v = 0$, and $(T-\lambda_3)\cdots(T-\lambda_n)v \neq 0$, then λ_2 is an eigenvalue of T; $\ldots$; keep arguing like this, if $(T-\lambda_{n-1})(T-\lambda_n)v = 0$ and $(T-\lambda_n)v \neq 0$, then λ_{n-1} is an eigenvalue; Otherwise if $(T-\lambda_n)v = 0$, then $\lambda_n$is an eigenvalue. In any case, we can have that $\exists\ 1 \leq i \leq n$, such that λ_i is an eigenvalue of T, where $p(\lambda_i) = \alpha$.
$(\Leftarrow)$: If there exists an eigenvalue λ of T, such that $p(\lambda) = \alpha$. Then by Question 10, we have $p(T)v = p(\lambda)v = \alpha v$, which implies that α is an eigenvalue of $p(T)$.

20 Denote $n = \dim V$. Then by Theorem 5.27, there exist a basis $\alpha = \{e_1, e_2, \ldots, e_n\}$ of V, such that

$$Te_k \in \text{span}(\{e_1, e_2, \ldots, e_k\}), \ \forall\ 1 \leq k \leq n.$$

So for any $1 \leq k \leq n$, the space $\text{span}(\{e_1, e_2, \ldots, e_k\})$ is a $k-$dimensional invariant subspace of V with respect to T.

Exercise 5.C

6 Denote $n = \dim V$. Since T has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, then the corresponding eigenvectors $v_1, \ldots, v_n$ generate a basis of V. Thus, $v_1, \ldots, v_n$ are also eigenvectors of S, then there exist $\mu_1, \ldots, \mu_n$, such that $Sv_i = \mu_i v_i, \ \forall\ 1 \leq i \leq n$. So we have that for $\forall\ 1 \leq i \leq n$,

$$\begin{aligned} STv_i &= S(\lambda_i v_i) = \lambda_i S v_i = \lambda_i \mu_i v_i \\ TSv_i &= T(\mu_i v_i) = \mu_i T v_i = \mu_i \lambda_i v_i. \end{aligned}$$

Thus $STv_i = TSv_i, \ \forall\ 1 \leq i \leq n$. And combined with that $v_1, \ldots, v_n$ is a basis of V, then $ST = TS$.

12 Denote $(\lambda_1, \lambda_2, \lambda_3) = (2, 6, 7)$. Since the eigenvalues are distinct, we can take a basis $v_1, v_2, v_3 \in \mathbf{F}^3$ and a basis $w_1, w_2, w_3 \in \mathbf{F}^3$ to be the eigenvectors of R and T respectively.
Define

$$S : V \longrightarrow V$$

$$\text{via} \ \ Sv_i = w_i, \ \forall\ 1 \leq i \leq n, \ \text{and} \ S(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i w_i$$

then $S \in \mathcal{L}(V)$ and is invertible. And we have

$$S^{-1}TSv_i = S^{-1}Tw_i = \lambda_i S^{-1} w_i = \lambda_i v_i = Rv_i, \ \forall\ 1 \leq i \leq 3.$$

which implies that $S^{-1}TS = R$.

16 (a). Using the inductive method, firstly we have $T(0,1) = (1, 0+1) = (1,1) = (F_1, F_2)$, so it holds for $n = 1$. Then we assume for any $1 \leq k \leq n$, there holds $T^k(0,1) = (F_k, F_{k+1})$, then we have

$$T^{n+1}(0,1) = T(T^n(0,1)) = T(F_n, F_{n+1}) = (F_{n+1}, F_n + F_{n+1}) = (F_{n+1}, F_{n+2}).$$

So by the inductive method, it holds for all $n \geq 1$.

(b). If λ is an eigenvalue, then there exists a nonzero vector (x, y), such that $T(x,y) = \lambda(x,y)$, then we have $y = \lambda x, \ x + y = \lambda y$. If $x = 0$, then $y = 0$, which is a contradiction. So $x \neq 0$ and by $x + \lambda x = \lambda^2 x$, we have $\lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \dfrac{1 \pm \sqrt{5}}{2}$. So T has two eigenvalues $\lambda_1 = \frac{1-\sqrt{5}}{2}$ and $\lambda_2 = \frac{1+\sqrt{5}}{2}$.

(c). By (b), we can have that for $\lambda_1 = \frac{1-\sqrt{5}}{2}, y = \frac{1-\sqrt{5}}{2}x$, then the eigenspace $E(\frac{1-\sqrt{5}}{2}, T) = \text{span}((1, \frac{1-\sqrt{5}}{2}))$. Similarly, we can have $E(\frac{1+\sqrt{5}}{2}, T) = \text{span}((1, \frac{1+\sqrt{5}}{2}))$.
So $\{(1, \frac{1-\sqrt{5}}{2}), (1, \frac{1+\sqrt{5}}{2})\}$ is a basis of $\mathbb{R}^2$.

(d). Denote $e_1 = (1, \frac{1-\sqrt{5}}{2})$, $e_2 = (1, \frac{1+\sqrt{5}}{2})$. Note that $(0,1) = \frac{1}{\sqrt{5}}(e_2 - e_1)$, then

$$\begin{aligned} T(0,1) &= \frac{1}{\sqrt{5}}(Te_2 - Te_1) = \frac{1}{\sqrt{5}}(\lambda_2 e_2 - \lambda_1 e_1) \\ \Rightarrow\ T^2(0,1) &= \frac{1}{\sqrt{5}}(\lambda_2 Te_2 - \lambda_1 Te_1) = \frac{1}{\sqrt{5}}(\lambda_2^2 e_2 - \lambda_1^2 e_1) \\ \Rightarrow\ \cdots \Rightarrow\ T^n(0,1) &= \frac{1}{\sqrt{5}}(\lambda_2^n e_2 - \lambda_1^n e_1) \end{aligned}$$

By (a), we have $(F_n, F_{n+1}) = \frac{1}{\sqrt{5}}(\lambda_2^n e_2 - \lambda_1^n e_1)$, which implies that $F_n = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right)$.

(e). Denote $\alpha = \frac{1+\sqrt{5}}{2}$, then $0 < \alpha^{-1} = \frac{\sqrt{5}-1}{2} < 1$. Then $F_n = \frac{1}{\sqrt{5}}(\alpha^n - (-\alpha)^{-n})$.

If n is even, then $F_n = \frac{1}{\sqrt{5}}(\alpha^n - \alpha^{-n}) < \frac{1}{\sqrt{5}}\alpha^n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$.
While $F_n + 1 - \frac{1}{\sqrt{5}}\alpha^n = \frac{1}{\sqrt{5}}(\sqrt{5} - \alpha^{-n}) > 0$, by $0 < \alpha^{-1} = \frac{\sqrt{5}-1}{2} < 1$. So it remains to prove that $F_n + 1 - \frac{1}{\sqrt{5}}\alpha^n > \frac{1}{\sqrt{5}}\alpha^n - F_n$, which is equivalent to prove that $F_n > \frac{1}{\sqrt{5}}\alpha^n - \frac{1}{2}$, which is equivalent to prove $\alpha^{-n} < \frac{\sqrt{5}}{2}$, which obviously holds since the $0 < \alpha^{-1} = \frac{\sqrt{5}-1}{2} < 1$ and $\sqrt{5} > 2$.
If n is odd, then $F_n = \frac{1}{\sqrt{5}}(\alpha^n + \alpha^{-n}) > \frac{1}{\sqrt{5}}\alpha^n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n$.
While $F_n - 1 - \frac{1}{\sqrt{5}}\alpha^n = \frac{1}{\sqrt{5}}(\alpha^{-n} - 1) < 0$. So it remains to prove that $F_n - \frac{1}{\sqrt{5}}\alpha^n < \frac{1}{\sqrt{5}}\alpha^n - F_n + 1$, which is equivalent to prove that $F_n < \frac{1}{\sqrt{5}}\alpha^n + \frac{1}{2}$, which is equivalent to prove $\alpha^{-n} < \frac{\sqrt{5}}{2}$, which obviously holds.

Extra Question

I Firstly we find the eigenvalues of T. Assume there exists a nonzero polynomial $p(x) = a_2x^2 + a_1x + a_0$ such that $T(p) = \lambda p(x)$ for some $\lambda \in \mathbb{R}$. Then we have

$$(2a_2 + a_1)(x^2 - 5x) = (\lambda + 1)(a_2x^2 + a_1x + a_0)$$

$$\Rightarrow \begin{cases} a_0(\lambda+1) = 0 \\ a_1(\lambda+1) = -5(2a_2 + a_1) \\ a_2(\lambda+1) = 2a_2 + a_1 \end{cases} \Rightarrow \begin{bmatrix} \lambda+1 & 0 & 0 \\ 0 & \lambda+6 & 10 \\ 0 & -1 & \lambda-1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since $p(x)$ is nonzero, we have that the column $(a_0, a_1, a_2)^T$ should be nonzero, which means the determinant of the matrix must be 0, so we have

$$(\lambda+1)\left[(\lambda+6)(\lambda-1)+10\right] = 0 \ \Rightarrow\ \lambda = -1 \text{ or } -4.$$

For $\lambda = -1$, we have that $2a_2 + a_1 = 0 \ \Rightarrow\ p(x) = a_2x^2 - 2a_2x + a_0$ is an eigenvector, which implies that the eigenspace $E(-1, T) =$span $(\{x^2 - x, 1\})$.
For $\lambda = -4$, we have that $a_0 = 0, 5a_2 + a_1 = 0 \ \Rightarrow\ p(x) = a_2x^2 - 5a_2x$ is an eigenvector, which implies that the eigenspace $E(-4, T) =$span $(\{x^2 - 5x\})$.
Since $\mathcal{P}(\mathbb{R})$ is 3-dimensional, and $x^2 - x, 1, x^2 - 5x$ is linearly independent, we have that $\{x^2 - x, 1, x^2 - 5x\}$ is a basis of $\mathcal{P}(\mathbb{R})$. By Theorem 5.41, T is diagonalizable.

II Similar to [I], we first compute the eigenvalues of T. Assume there exists a nonzero polynomial $p(x) = a_2x^2 + a_1x + a_0$ such that $T(p) = \lambda p(x)$ for some $\lambda \in \mathbb{R}$. Then we have

$$2a_2x^2 + 2a_1x + 2a_0 + (x^2 - 9)(9a_2 + 3a_1 + a_0) = \lambda(a_2x^2 + a_1x + a_0)$$

$$\Rightarrow \begin{cases} 2a_2 + 9a_2 + 3a_1 + a_0 = \lambda a_2 \\ 2a_1 = \lambda a_1 \\ 2a_0 - 9(9a_2 + 3a_1 + a_0) = \lambda a_0 \end{cases} \Rightarrow \begin{bmatrix} 1 & 3 & 11-\lambda \\ 0 & 2-\lambda & 0 \\ \lambda+7 & 27 & 81 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Since the determinant of the matrix is zero, we have that

$$(2-\lambda)[81 - (11-\lambda)(\lambda+7)] = 0 \ \Rightarrow \ \lambda = 2.$$

So T has only one eigenvalue $\lambda = 2$, which implies that $9a_2 + 3a_1 + a_0 = 0 \ \Rightarrow$ the eigenvector $p(x) = a_2x^2 + a_1x - 3a_1 - 9a_2 = a_2(x^2 - 9) + a_1(x - 3)$. So the eigenvectors of T are in the eigenspace $E(2, T) = \text{span}(\{x^2 - 9, x - 3\})$, which cannot generate a basis of T. By Theorem 5.41, T is not diagonalizable.