Math 2040C Week 7

Linear Operator

Defn 3.67 A linear map $T: V \to V$ is called a linear operator on V.

Notations: $L(V) = L(V,V)$

$$M(T,\alpha) = M(T,\alpha,\alpha)$$

where α is an ordered basis of V.

Prop 3.69 If $\dim V < \infty$, then $T \in L(V)$ is invertible $\iff$ injective $\iff$ surjective

Pf Follows easily from

$$\dim V = \dim \operatorname{null} T + \dim \operatorname{range} T$$

(Exercise)

Diagonalization (Chapter 5)

Goal: To understand linear operators better

Given $T \in L(V)$, want a decomposition

$$V = U_1 \oplus \cdots \oplus U_m$$

so that the restrictions $T|_{U_i} : U_i \longrightarrow V$ are easy to understand. Nice if $T|_{U_i}$ are operators

Defn 5.2 Suppose $T \in L(V)$. A subspace $U \subseteq V$ is called T-invariant (invariant under T) if $T(u) \in U$ for all $u \in U$.

Rmk If U is T-invariant, then $T|_U \in L(U)$

eg For any $T \in L(V)$,

$\{0\}$, V, $\operatorname{null} T$, $\operatorname{range} T$ are T-invariant.

Q What are 1-dim T-invariant subspaces?

1 dim $\rightsquigarrow$ $\operatorname{span}\{v\}$, $v \neq \vec{0}$

T-invariant $\Rightarrow T(v) \in \operatorname{span}\{v\}$

$\Rightarrow T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$.

Defn 5.5, 5.7

Let $T \in L(V)$. Suppose $T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$ and $v \in V$ with $v \neq \vec{0}$.

Then λ is called an eigenvalue of T,

v is called an eigenvector corresponding to λ

Rmk We write e.value for eigenvalue

e.vector for eigenvector

Note $T(v) = \lambda v \Leftrightarrow T(v) - \lambda v = \vec{0}$

$\Leftrightarrow T(v) - \lambda I_V(v) = \vec{0}$

$\Leftrightarrow (T - \lambda I_V)(v) = \vec{0}$

"Prop 5.6" Let $\dim V < \infty$, $T \in L(V)$, $\lambda \in \mathbb{F}$

Then TFAE (The followings are equivalent)

1. λ is an e.value
2. $T - \lambda I_V$ is not injective
3. $T - \lambda I_V$ is not surjective
4. $T - \lambda I_V$ is not bijective
5. $M(T - \lambda I_V, \beta)$ is not invertible $\leftarrow$ for any ordered basis β of V
6. $\det M(T - \lambda I_V, \beta) = 0$ $\longleftarrow$

Rmk 5, 6 are not from textbook

Write $M(T) = M(T, \beta)$, we have

$$\det M(T - \lambda I) = \det[M(T) - \lambda M(I_V)]$$
$$\det(M(T) - \lambda I)$$

Note $p(t) := \det(M(T) - tI)$ is the characteristic polynomial of $M(T)$

$$\boxed{\lambda \text{ is an e.value of } T \Leftrightarrow p(\lambda) = 0}$$

eg Let $T:\mathbb{F}^2 \to \mathbb{F}^2$, $T(x,y)=(-y,x)$

Find eigenvalues and eigenvectors if

① $\mathbb{F}=\mathbb{R}$ ② $\mathbb{F}=\mathbb{C}$.

<u>Sol</u> <u>Method 1</u>

If $T(x,y)=\lambda(x,y)$, then

$$(-y,x)=(\lambda x,\lambda y) \Rightarrow \begin{cases} -y=\lambda x \\ x=\lambda y \end{cases}$$

$$\Rightarrow -y=\lambda x=\lambda^2 y \Rightarrow (\lambda^2+1)y=0 \quad (*)$$

① If $\lambda\in\mathbb{R}$, then $\lambda^2+1\neq 0 \Rightarrow y=0$

$\therefore x=\lambda y=0 \Rightarrow (x,y)=(0,0)$

$\therefore$ No eigenvalue/eigenvector if $\mathbb{F}=\mathbb{R}$

② If $\lambda\in\mathbb{C}$, then $(*) \Rightarrow \lambda^2=-1$ or $y=0$

If $y=0$, then $x=\lambda y=0 \Rightarrow \underbrace{(x,y)=(0,0)}_{\text{not eigenvector}}$

If $\lambda^2=-1$, then $\lambda=\pm i$.

<u>For $\lambda=i$</u>

$x=iy \Rightarrow (x,y)=(iy,y)=y(i,1)$

e.vectors are $y(i,1)$, $y\in\mathbb{C}\setminus\{0\}$

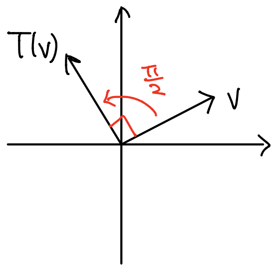
<u>For $\lambda=-i$</u>

$x=-iy \Rightarrow (x,y)=(-iy,y)=y(-i,1)$

e.vectors are $y(-i,1)$, $y\in\mathbb{C}\setminus\{0\}$

<u>Rmk</u> ① Existence of e.values depends on $\mathbb{F}$

② If $\mathbb{F}=\mathbb{R}$, T is rotation around origin in $\mathbb{R}^2$ by $\frac{\pi}{2}$ anti-clockwisely



Geometrically, for $v\neq 0$

$T(v)$ and v are not in

same or opposite direction

$\Rightarrow T(v)\neq\lambda v$ for any $\lambda\in\mathbb{R}$

Method 2 : Matrix (Usually better)

Let $\beta = \{e_1, e_2\}$ and $A = M(T, \beta)$

Then $T(e_1) = T(1,0) = (0,1) = 0e_1 + 1e_2$

$T(e_2) = T(0,1) = (-1,0) = (-1)e_1 + 0e_2$

$\therefore A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, which has char poly

$\det(A - tI) = \begin{vmatrix} -t & -1 \\ 1 & -t \end{vmatrix} = t^2 + 1 = (t+i)(t-i)$

$\therefore$ eigenvalues of A (and T) are $\lambda_1 = i$, $\lambda_2 = -i$ (if $\mathbb{F} = \mathbb{C}$)

For λ_1, $A - \lambda_1 I = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}$

$\left[\begin{array}{cc|c} -i & -1 & 0 \\ 1 & -i & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = t \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\therefore$ e.vector of A corr. to λ_1 are $t\begin{bmatrix} i \\ 1 \end{bmatrix}$, $t \in \mathbb{C} \setminus \{0\}$

e.vector of T corr to λ_1 are

$tie_1 + te_2 = t(i, 1)$, Same as by method 1

Similar calculation for $\lambda_2 = -i$

Q Let $D: C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$, $Df = f'$

What are eigenvalues and eigenvectors?

A $Df = \lambda f \Rightarrow f' = \lambda f$

Note $(e^{\lambda t})' = \lambda e^{\lambda t}$

$\therefore e^{\lambda t}$ is an e.vector corr to e.value λ

Fact Each "eigenspace" has dim 1

$\therefore$ Every $\lambda \in \mathbb{R}$ is an eigenvalue

and corresponding eigenvectors are

$Ce^{\lambda t}$ where $C \neq 0$

Properties of eigenvalues / eigenvector

> **Prop 5.10** Let $T \in L(V)$,
> $\lambda_1, \cdots, \lambda_m$ be distinct eigenvalues of T
> v_i be an eigenvector corresponding to λ_i
> Then $v_1, \cdots, v_m$ are linearly independent

Pf We prove by contradiction.

Suppose $v_1, \cdots, v_m$ are linearly dept.

Let k be the smallest integer s.t.

$$v_k \in \operatorname{span}\{v_1, \cdots, v_{k-1}\}$$

(By linear dependence lemma, k exists)

Let $v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1}$ (i)

$\Rightarrow T(v_k) = a_1 T(v_1) + \cdots + a_{k-1} T(v_{k-1})$

$\lambda_k v_k = a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1}$ (ii)

(i) $\times \lambda_k$ − (ii) :

$0 = a_1(\lambda_k - \lambda_1) v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1}) v_{k-1}$

Note v_k is e.vector $\Rightarrow v_k \neq \vec{0}$

$\Rightarrow$ some $a_i \neq 0$ by (i)

$\Rightarrow a_i(\lambda_k - \lambda_i) \neq 0$

$\therefore v_1, \cdots, v_{k-1}$ are lin dept

By linear dependence lemma, $\exists j \leq k-1$ s.t.

$$v_j \in \operatorname{span}\{v_1, \cdots, v_{j-1}\}$$

A contradiction to the choice of k

> **Cor 5.13** Let $\dim V = n < \infty$, $T \in L(V)$
> Then T has at most n distinct eigenvalues

Pf Suppose $\lambda_1, \cdots, \lambda_m$ are distinct e.values of T and $v_1, \cdots, v_m$ be corr. e.vectors.

By Prop 5.10, $v_1, \cdots, v_m$ are lin indept.

$\Rightarrow m \leq \dim V$

Polynomials of operator

If $S, T \in L(V)$, then $ST = S \circ T$ is defined.

Composition $\rightsquigarrow$ Product structure on $L(V)$

Defn 5.16 Let $m \in \mathbb{N}$, $T \in L(V)$. Define

- $T^m = \underbrace{T \cdot T \cdots T}_{m \text{ times}}$
- $T^0 = I_V$
- $T^{-m} = (T^{-1})^m$ if T is invertible

Defn 5.17 Let $T \in L(V)$, $p \in P(\mathbb{F})$ with

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

Define $p(T) \in L(V)$ to be

$$p(T) = a_0 I_V + a_1 T + \cdots + a_m T^m$$

Prop 5.20 Let $p, q \in P(\mathbb{F})$, $T \in L(V)$. Then

1. $(pq)(T) = p(T)q(T)$
2. $p(T)q(T) = q(T)p(T)$

eg. Let $p(z) = 1 + z + z^2$ $\quad q(z) = 1 - z$

Then
$$\begin{aligned} p(T)q(T) &= (I_V + T + T^2)(I_V - T) \\ &= I_V + T + T^2 - T - T^2 - T^3 \\ &= I_V - T^3 \\ &= (pq)(T) \end{aligned}$$

Prop 5.21 Let $\mathbb{F} = \mathbb{C}$, $0 < \dim V < \infty$

A linear operator $T \in L(V)$ has an eigenvalue

Pf 1 Let β be any ordered basis of V

$p(t)$ be the char poly of $A = M(T, \beta)$

Then $\deg p(t) = \dim V > 0$

By Fundamental Theorem of algebra,

$p(t)$ has a root $\lambda \in \mathbb{C} \Rightarrow \lambda$ is an e.value

<u>Pf 2</u> Let $\dim V = n$, $v \in V$

The list $v, Tv, T^2v, \cdots, T^n v$

has length $n+1 \Rightarrow$ linearly dependent

$\Rightarrow \exists\, a_0, \cdots, a_n \in \mathbb{C}$, not all 0, s.t.

$$a_0 v + a_1 T v + \cdots + a_n T^n v = 0$$

Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ $\quad m = \deg p$

$$= a_m (z - \alpha_1) \cdots (z - \alpha_m) \quad a_m \neq 0$$

$\Rightarrow a_m (T - \alpha_1 I_V) \cdots (T - \alpha_m I_V)(v) = p(T)(v) = \vec{0}$

$\because \vec{v} \neq 0$

$\Rightarrow a_m (T - \alpha_1 I_V) \cdots (T - \alpha_m I_V)$ is not injective

$\Rightarrow$ Some $T - \alpha_i I_V$ is not injective

$\Rightarrow \alpha_i$ is an eigenvalue

Given $T \in L(V)$, can we find α st. $M(T, \alpha)$ is diagonal? No Upper triangular? yes if $\mathbb{F} = \mathbb{C}$

<u>Prop 5.26</u> Suppose $T \in L(V)$ $\alpha = \{v_1, v_2, \cdots, v_n\}$ is an ordered basis of V. Then TFAE

1. $M(T, \alpha)$ is upper triangular
2. $T(v_j) \in \operatorname{span}\{v_1, \cdots v_j\}$ for each $j = 1, 2, \cdots n$
3. $\operatorname{Span}\{v_1, \cdots, v_j\}$ is T-invariant for each $j = 1, 2, \cdots n$

<u>Pf</u> 2 $\Rightarrow$ 1 : Suppose 2, then for each j

$$T(v_j) = a_{1j} v_1 + \cdots + a_{jj} v_j + 0 \cdot v_{j+1} + \cdots + 0 \cdot v_n$$

$$\Rightarrow j\text{-th column of } M(T, \alpha) = \begin{array}{cl} \left[\begin{array}{c} a_{1j} \\ \vdots \\ a_{jj} \\ 0 \\ \vdots \\ 0 \end{array}\right] & \begin{array}{l} \\ \\ \leftarrow j\text{-th row} \\ \leftarrow (j+1)\text{-th row} \\ \\ \\ \end{array} \end{array}$$

$\Rightarrow$ 1

Pf of the rest : Ex

eg let $T: P_n(\mathbb{R}) \to P_n(\mathbb{R})$

$$Tp = p + p'$$

Let $\beta = \{1, x, x^2, \cdots, x^n\}$

then $\mathrm{span}\{1, x, \cdots x^j\} = P_j(\mathbb{R})$

is clearly T-invariant

$\therefore$ $M(T, \beta)$ is upper triangular

If $n = 4$, then

$$M(T,\alpha) = \begin{bmatrix} 1 & 1 & & & \\ & 1 & 2 & & \\ & & 1 & 3 & \\ & 0 & & 1 & 4 \\ & & & & 1 \end{bmatrix}$$

$\because T(x^j) \in \mathrm{span}\{1, \cdots, x^j\}$

Prop 5.27 Let $T \in \mathcal{L}(V)$, $\dim V < \infty$, $\mathbb{F} = \mathbb{C}$

Then $\exists$ ordered basis β s.t.

$M(T, \beta)$ is upper triangular

Pf We prove by induction on $\dim V = n$

- The statement is clearly true for $n=1$
- Assume true for $n \leq k$

  Let $\dim V = n = k+1$

By 5.21, $T \in \mathcal{L}(V)$ has an eigenvalue λ

Let $U = \text{range}(T - \lambda I_V)$

$T - \lambda I_V$ is not surjective $\Rightarrow \dim U \leq \dim V - 1 = k$

Also, if $u \in U$, then

$$T(u) = \underbrace{(T - \lambda I_V)u}_{\text{Both are in } U} + \underbrace{\lambda u}_{} \in U$$

$\Rightarrow U$ is T-invariant

Hence, $T|_U \in \mathcal{L}(U)$, $\dim U \leq k$

Induction assumption

$\Rightarrow \exists$ ordered basis $\alpha = \{u_1, \cdots, u_m\}$ of U s.t.

$M(T|_U : \alpha)$ is upper triangular

Extend α to an ordered basis of V

$\beta = \{u_1, \cdots, u_m, v_{m+1}, \cdots, v_n\}$

Then $T(v_j) = (T - \lambda I_V) v_j + \lambda v_j$

Note $(T - \lambda I_V) v_j \in U = \text{span}\, \alpha$

$$\therefore M(T, \beta) = \begin{array}{c} \left[\begin{array}{c:c} M(T|_U, \alpha) & * \\ \hdashline 0 & \lambda I_{n-m} \end{array} \right] \begin{array}{l} \}\, m \\ \}\, n-m \end{array} \\ \begin{array}{cc} \underbrace{\qquad\qquad}_{m} & \underbrace{\qquad}_{n-m} \end{array} \end{array}$$

is upper triangular

Prop 5.30 Let $T \in L(V)$, $M(T,\alpha)$ is upper triangular. then

T is invertible $\Leftrightarrow$ All diagonal entries of $M(T,\alpha)$ are non-zero

Pf

Let $M(T,\alpha) = \begin{bmatrix} a_{11} & & * \\ & a_{22} & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix}$

T is invertible

$\Leftrightarrow$ $M(T,\alpha)$ is invertible

$\Leftrightarrow$ $\det M(T,\alpha) = a_{11}a_{22}\cdots a_{nn} \neq 0$

$\Leftrightarrow$ All diagonal entries $a_{ii} \neq 0$

Rmk The proof in textbook does not use determinant.

Prop 5.32

Suppose $M(T,\alpha)$ is upper-triangular, then eigenvalues of T are the diagonal entries of $M(T,\alpha)$

Pf Let $M(T,\alpha)$ be the same as in last proof

λ is an eigenvalue

$\Leftrightarrow$ $T - \lambda I$ is not invertible

$\Leftrightarrow$ $M(T-\lambda I_V,\alpha) = \begin{bmatrix} a_{11}-\lambda & & * \\ & a_{22}-\lambda & \\ & & \ddots & \\ 0 & & & a_{nn}-\lambda \end{bmatrix}$

is not invertible

$\Leftrightarrow$ $a_{ii} - \lambda = 0$ for some i (Prop 5.30)

$\Leftrightarrow$ λ is a diagonal entry of $M(T,\alpha)$