Math 2040C Week 3

Let V be a vector space over $\mathbb{F}$

Span

Def A *linear combination* of $v_1, v_2, \cdots, v_m \in V$ is a vector of the form

$$\sum_{i=1}^{m} a_i v_i = a_1 v_1 + a_2 v_2 + \cdots + a_m v_m$$

where $a_1, a_2, \cdots, a_m \in \mathbb{F}$.

Def Let $S \subseteq V$ be a subset.

If $S \neq \phi$, define the *span of S* to be

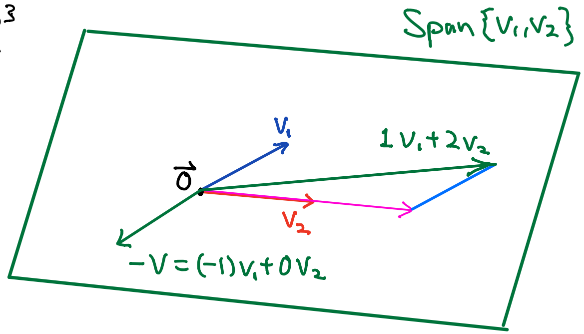
$$\operatorname{span} S = \left\{ \sum_{i=1}^{m} a_i v_i : a_i \in \mathbb{F}\ v_i \in S, \forall i = 1, 2, \cdots, m \right\}$$

= the set of all lin. combinations of vectors from S

If $S = \phi$, define $\operatorname{span} \phi = \{\vec{0}\}$

Pictures

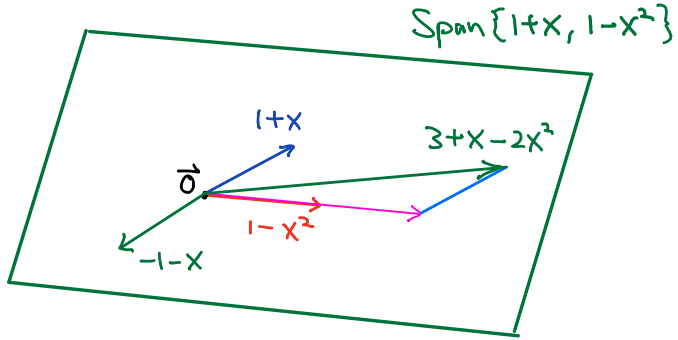
① $\mathbb{R}^3$



$-v_1$, $1v_1 + 2v_2$ are linear combinations of v_1, v_2

$\operatorname{Span}\{v_1, v_2\} = \operatorname{Span}\{v_1, v_2, v_1 + 2v_2\}$ is a plane

② $P(\mathbb{R})$



eg Let $V = P(\mathbb{R})$ and $S = \{1+x+x^2, 1+3x\}$

Determine whether $1-x+2x^2, 2+3x+x^2 \in \operatorname{Span} S$

Sol For $1-x+2x^2$, suppose

$$\begin{aligned} 1-x+2x^2 &= a_1(1+x+x^2) + a_2(1+3x) \\ &= (a_1+a_2) + (a_1+3a_2)x + a_1x^2 \end{aligned}$$

Comparing coefficients $\Rightarrow \begin{cases} a_1 + a_2 = 1 \\ a_1 + 3a_2 = -1 \\ a_1 \qquad = 2 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 2 & -2 \\ 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} a_2 = -1 \\ a_1 = 1 - a_2 = 2 \end{array}$$

$$\therefore 1-x+2x^2 = 2(1+x+x^2) + (-1)(1+3x)$$

$$\in \operatorname{span} S$$

Similarly, suppose

$$\begin{aligned} 2+3x+x^2 &= a_1(1+x+x^2) + a_2(1+3x) \\ &= (a_1+a_2) + (a_1+3a_2)x + a_1x^2 \end{aligned}$$

Comparing coefficients $\Rightarrow \begin{cases} a_1 + a_2 = 2 \\ a_1 + 3a_2 = 3 \\ a_1 \qquad = 1 \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & -1 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & -1 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{array}\right] \leftarrow \text{no solution}$$

$\therefore$ no such a_1, a_2 exist

$\Rightarrow 2+3x+x^2 \notin \operatorname{span} S$

eg

① Let $V = M_{2\times 2}(\mathbb{R})$. Then

$$\mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

= subspace of all symmetric 2×2 matrices

② $P(\mathbb{R}) = \mathrm{span}\{1, x, x^2, x^3, \cdots\}$

$P_m(\mathbb{R}) = \mathrm{span}\{1, x, x^2, \cdots, x^m\}$

③ Let $V = \mathbb{R}^\infty = \{(x_1, x_2, \cdots) : x_i \in \mathbb{R}\ \forall i \in \mathbb{N}\}$

$e_i = (0, 0, \cdots 0, \underset{\uparrow}{1}, 0, 0 \cdots)$ and $S = \{e_i : i \in \mathbb{N}\}$

i-th entry

Then

$\mathrm{span}\, S$ = the set of all $\vec{x} \in \mathbb{R}^\infty$ with finitely many non-zero terms

$= \{(x_1, x_2, \cdots) \in \mathbb{R}^\infty : \exists m \in \mathbb{N} \text{ s.t. } x_i = 0\ \forall i > m\}$

Rmk For ③, even though S is infinite, each linear combination in $\mathrm{Span}\, S$ is the sum of finitely many scalar multiples of vectors from S

$$(1, 2, 3, 4, 5, \underbrace{0, 0, 0, \cdots}_{\text{all zero}}) = e_1 + 2e_2 + \cdots + 5e_5 \in \mathrm{Span}\, S$$

but $\vec{a} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots) \notin \mathrm{Span}\, S$

Indeed, in a vector space V

$v_1 + v_2$ is defined for any $v_1, v_2 \in V$

Induction $\Rightarrow$ any finite sum $\sum_{i=1}^{m} v_i$ is defined

However, infinite sum cannot be defined without additional structure on V (eg. distance, limit ...)

Hence we cannot write

$$\vec{a} = e_1 + \frac{1}{2}e_2 + \frac{1}{3}e_3 + \cdots \quad \times$$

Prop 2.7 Let $S \subseteq V$ be a subset. Then $\operatorname{span} S$ is the smallest subspace of V containing S.

i.e. if $W \subseteq V$ is a subspace and $S \subseteq W$,

then $\operatorname{span} S \subseteq W$

Pf If $S = \phi$, then $\operatorname{span} S = \{0\}$

The proposition is clearly true.

Suppose $S \neq \phi$. Take any $v \in V$. Then

$$\vec{0} = 0v \in \operatorname{span} S$$

Also, for any $v, w \in \operatorname{span} S$, we have

$$v = a_1 v_1 + \cdots + a_m v_m,$$
$$w = b_1 v_1 + \cdots + b_n v_n$$

for some $a_i, b_j \in \mathbb{F}$, $v_i, w_j \in S$.

$\therefore\ v + w = a_1 v_1 + \cdots + a_m v_m + b_1 v_1 + \cdots + b_n v_n$

$\Rightarrow\ v + w \in \operatorname{span} S$

Similarly, if $\lambda \in \mathbb{F}$, then

$$\lambda v = \lambda (a_1 v_1 + \cdots + a_m v_m)$$
$$= (\lambda a_1) v_1 + \cdots + (\lambda a_m) v_m \in \operatorname{span} S$$

$\therefore\ \operatorname{span} S$ is a subspace of V

Clearly $S \subseteq \operatorname{span} S$ $\quad (v \in S \Rightarrow v = (1)\, v \in \operatorname{span} S)$

For "smallest", suppose $W \subseteq V$ is a subspace and $S \subseteq W$. Let $v \in \operatorname{span} S$.

Then $\exists\, v_1, \cdots, v_m \in S$, $a_1, \cdots, a_m \in \mathbb{F}$, such that

$$v = a_1 v_1 + \cdots + a_m v_m$$

$S \subseteq W \ \Rightarrow\ v_1, v_2, \cdots, v_m \in W$

$\Rightarrow\ v = a_1 v_1 + \cdots + a_m v_m \in W$ $\quad (\because W \text{ is a subspace})$

$\therefore\ \operatorname{span} S \subseteq W$

Ex $S \subseteq V$ is a subset. Show that

S is a subspace $\Leftrightarrow$ $S = \operatorname{span} S$

Defn 2.10 V is called **finite-dimensional** if $\exists$ finite subset $S \subseteq V$ such that $V = \operatorname{span} S$. Otherwise V is called **infinite-dimensional**.

eg $P_m(\mathbb{F})$ and $P(\mathbb{F})$

① $P_m(\mathbb{F}) = \operatorname{span}\{1, x, x^2, \cdots, x^m\}$

$\therefore$ $P_m(\mathbb{F})$ is finite dimensional

② $P(\mathbb{F})$? For any finite subset

$S = \{p_1(x), p_2(x), \cdots, p_n(x)\} \subseteq P(\mathbb{F})$

Let $m > \deg p_i$ for $1 \le i \le n$

Then $x^m \in P(\mathbb{F})$ but $x^m \notin \operatorname{span} S$

$\therefore$ $P(\mathbb{F}) \neq \operatorname{span} S$ for any finite subset S

$\therefore$ $P(\mathbb{F})$ is infinite dimensional

Ex Show $\mathbb{F}^n$, $M_{n\times n}(\mathbb{F})$ have finite dim

$\mathbb{F}^\infty$, $\mathbb{R}^{(0,1)}$ have infinite dim

Linear Independence

Suppose $v \in \operatorname{span}\{v_1, v_2, \cdots, v_n\}$. Then $\exists a_i \in \mathbb{F}$ such that $v = a_1 v_1 + \cdots + a_n v_n$

It is nice if such $a_1, a_2, \cdots, a_n$ are unique

Defn 2.18, 2.19

$v_1, \cdots v_n \in V$ are called **linearly independent** if

$$a_1 v_1 + \cdots + a_n v_n = \vec{0}.$$

only when $a_1 = \cdots = a_n = 0$

Otherwise, $v_1, \cdots, v_n$ are called **linearly dependent**

Rmk ① In other words, $v_1, \cdots, v_n$ are linearly dependent if $\exists a_1, \cdots, a_n \in \mathbb{F}$, not all zero, such that $a_1 v_1 + \cdots + a_n v_n = \vec{0}$.

② v is lin indept $\Leftrightarrow$ $v \neq \vec{0}$

v, w are lin. indept. $\Leftrightarrow$ $v \neq aw$ and $w \neq bw$ for any $a, b \in \mathbb{F}$

In our textbook, linear independence is defined on a list of vectors $v_1, \cdots v_n$ as in Defn 2.18.

It is indeed more common to define linear independence on a set of vectors:

Defn ⊛

A subset $S \subseteq V$ is called **linearly independent** for any **distinct** $v_1, \cdots, v_n \in S$

$$a_1 v_1 + \cdots + a_n v_n = \vec{0}$$

only when $a_1 = \cdots = a_n = 0$.

Otherwise, S is called **linearly dependent**.

Rmk

① The empty set ϕ is linearly independent

② S can be finite or infinite in Defn ⊛

③ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is lin dept (Defn 2.18)

$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is lin indept (Defn ⊛)

④ If $v_1, \cdots, v_n \in V$ are **distinct**, then

$v_1, \cdots, v_n$ are lin. indept (as in Defn 2.18) $\iff$ $\{v_1, \cdots, v_n\}$ is lin. indept (as in Defn ⊛)

Prop Let $S_1 \subseteq S_2 \subseteq V$.

① S_1 is lin dept $\Rightarrow$ S_2 is lin dept

② S_2 is lin indept $\Rightarrow$ S_1 is lin indept.

③ $\operatorname{span} S_1 \subseteq \operatorname{span} S_2$

Prop Suppose $S \subseteq V$ is a subset. Then

S is lin indept $\iff$

Every $v \in \operatorname{span} S$ can be expressed as a linear combination of vectors from S uniquely

Ex Prove the propositions

eg Is $S=\{1+2x, 3-x+x^2, 1-5x+x^2\}$ linearly independent?

Sol Suppose

zero polynomial

$$a_1(1+2x)+a_2(3-x+x^2)+a_3(1-5x+x^2)=0$$

$$a_1+3a_2+a_3+(2a_1-a_2-5a_3)x+(a_2+a_3)x^2=0$$

$$\Leftrightarrow \begin{cases} a_1+3a_2+a_3=0 \\ 2a_1-a_2-5a_3=0 \\ a_2+a_3=0 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & -1 & -5 & 0 \\ & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & 3 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

free variable a_3

$\therefore$ ∞ many solutions. In particular

Put $a_3=1$, then $a_2=-1$, $a_1=2$

$$(1)(1+2x)+(-1)(3-x+x^2)+(1)(1-5x+x^2)=0$$

and S is lin. dept

eg Let $S=\{\cos x, \cos 2x, \cos 3x\} \subseteq \mathbb{R}^{\mathbb{R}}$

Is S lin. indept?

Sol Suppose

zero function

$$a_1\cos x + a_2 \cos 2x + a_3 \cos 3x = 0 \quad (\forall x \in \mathbb{R})$$

Put $x=0, \frac{\pi}{3}, \frac{\pi}{2}$

$$\Rightarrow \begin{cases} a_1+a_2+a_3=0 \\ \frac{1}{2}a_1-\frac{1}{2}a_2-a_3=0 \\ -a_2=0 \end{cases}$$

Gaussian elimination $\Rightarrow a_1=a_2=a_3=0$

$\therefore$ S is lin indept.

Ex Show that

$\{e^{nx} : n\in\mathbb{Z}\} \subseteq \mathbb{R}^{\mathbb{R}}$ is lin indept

Hint: Differentiation and Vandermonde matrix

Lemma 2.21 (Linear Dependence Lemma)

Suppose $v_1, \cdots, v_m \in V$ are lin. dept. Then $\exists j \in \{1, \cdots, m\}$ such that

① $v_j \in \operatorname{span}\{v_1, v_2, \cdots, v_{j-1}\}$

② $\operatorname{span}\{v_1, \cdots, v_m\} = \operatorname{span}(\{v_1, \cdots, v_m\} \setminus \{v_j\})$

Pf $v_1, v_2, \cdots, v_m$ are lin. dept.

$\Rightarrow \exists a_1, \cdots, a_m \in \mathbb{F}$, not all zero, such that

$$a_1 v_1 + \cdots + a_m v_m = \vec{0}$$

Let j be the largest integer such that $a_j \neq 0$.

Then $a_1 v_1 + \cdots + a_j v_j = \vec{0}$

$$v_j = -\frac{a_1}{a_j} v_1 - \frac{a_2}{a_j} v_2 - \cdots - \frac{a_{j-1}}{a_j} v_j$$

$\Rightarrow$ ①

For ②, let $S = \{v_1, \cdots, v_m\}$ and $v \in \operatorname{span} S$

Then $\exists b_1, \cdots, b_m \in \mathbb{F}$ such that

$$v = \sum_i b_i v_i = \left(\sum_{i \neq j} b_i v_i\right) + b_j v_j = \sum_{i \neq j} \left(b_i - \frac{a_i}{a_j}\right) v_i$$

$\Rightarrow v \in \operatorname{span}(S \setminus \{v_j\})$

$\therefore \operatorname{span} S \subseteq \operatorname{span}(S \setminus \{v_j\})$

Also $\operatorname{span}(S \setminus \{v_j\}) \subseteq \operatorname{span} S$ $(\because S \setminus \{v_j\} \subseteq S)$

$\therefore \operatorname{span}(S \setminus \{v_j\}) = \operatorname{span} S \Rightarrow$ ②

Prop Let $v_1, \cdots, v_n \in V$ be lin. indept.

Suppose $w \in V$. Then

$w \in \operatorname{span}\{v_1, \cdots, v_n\} \Leftrightarrow w, v_1, \cdots, v_n$ are lin dept.

Prop 2.23

Let $S_1, S_2 \subseteq V$ be finite subsets. Suppose

① S_1 is lin indept.

② $\operatorname{span} S_2 = V$

Then $|S_1| \leq |S_2|$ ($|S|$ = Number of elements in S)

Pf let $S_1 = \{u_1, \cdots, u_m\}$ has m vectors

$S_2 = \{v_1, \cdots, v_n\}$ has n vectors

We will prove $m \leq n$ by "Replacement algorithm" as follows.

① $u_1 \in V = \operatorname{span}\{v_1, \cdots, v_n\}$

$\Rightarrow u_1, v_1, \cdots, v_n$ are lin dept.

Note that S_1 is lin. indept $\Rightarrow u_1 \neq \vec{0}$

Lemma 2.21 $\Rightarrow$

$\exists j$ such that $v_j \in \operatorname{span}\{u_1, v_1, \cdots, v_{j-1}\}$

Also, $\operatorname{span}\{u_1, v_i : i \neq j\} = \operatorname{span}\{u_1, v_i\} = V$

i.e. we replace a vector v_j in S_2 by u_1 and the resulting subset still spans V

By re-ordering $v_1, \cdots, v_n$ if necessary, we assume $j = n$

$\Rightarrow \operatorname{span}\{u_1, v_1, \cdots v_{n-1}\} = V$

② By a similar argument and another re-ordering if necessary, we can replace v_{n-1} by u_2 and

$$\operatorname{span}\{u_1, u_2, v_1, \cdots, v_{n-2}\} = V$$

③ Repeat the process above.

If $m > n$, then after n times, we have

$$\operatorname{span}\{u_1, \cdots, u_n\} = V$$

Hence, $u_{n+1} \in V = \operatorname{span}\{u_1, \cdots, u_n\}$

But S_1 is lin indept, a contradiction

$\therefore m \leq n$

eg $S = \{1+x+x^2, 1-x, x-x^2, 1+7x^2\}$

Note ① $S \subseteq P_2(\mathbb{R})$

② $P_2(\mathbb{R}) = \text{span}\{\underbrace{1, x, x^2}_{\text{3 elements}}\}$

③ $|S| = 4 > 3$

Lemma 2.23 $\Rightarrow$ S is lin. dept.

> **Prop 2.26**
>
> Every subspace of a finite dimensional vector space is finite-dimensional

Pf let V be a finite dim vector space with $V = \text{span}\{v_1, \cdots, v_n\}$ and $W \leq V$ be a subspace

If $W = \{0\}$, then $W = \text{span}\,\phi$ is finite dim

If $W \neq \{0\}$, let $w_1 \in W$ and $w_1 \neq \vec{0}$.

If $W \neq \text{span}\{w_1\}$, choose $w_2 \in W \setminus \text{span}\{w_1\}$

If $W \neq \text{span}\{w_1, w_2\}$, choose $w_3 \in W \setminus \text{span}\{w_1, w_2\}$

We continue this process to choose

$w_{k+1} \in W \setminus \text{span}\{w_1, \cdots, w_k\}$

until $W = \text{span}\{w_1, \cdots, w_k\}$

Claim: $W = \text{span}\{w_1, \cdots, w_k\}$ for some $k \leq n$

Otherwise, by the process above

$\exists\, w_1, \cdots, w_{n+1} \in W$ such that

$w_j \notin \text{span}\{w_1, \cdots, w_{j-1}\}$ for any $1 \leq j \leq n+1$

Lemma 2.21 $\Rightarrow$ $w_1, \cdots, w_{n+1}$ are lin. indept

but $V = \text{span}\{v_1, \cdots, v_n\}$

Prop 2.23 $\Rightarrow$ $n+1 \leq n$, a contradiction

$\therefore$ The claim is true $\Rightarrow$ W is finite-dim

Basis

Defn A subset $S \subseteq V$ is called a basis of V if $V = \operatorname{span} S$ and S is linearly independent.

Examples of basis

① Let $e_i = (0, \cdots, 0, \overset{\text{i-th entry}}{1}, 0, \cdots, 0) \in \mathbb{F}^n$

$\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbb{F}^n$.

② $\{(1,2),(3,4)\} \subseteq \mathbb{R}^2$

③ $\{(1,2,0),(0,0,1)\} \subseteq \{(x,2x,y) \in \mathbb{R}^3 : x, y \in \mathbb{R}\}$

④ $\{1, x, x^2, \cdots x^m\} \subseteq P_m(\mathbb{F})$

⑤ $\{1, x, x^2, \cdots\} \subseteq P(\mathbb{F})$

⑥ Let $1 \le i, k \le m$, $1 \le j, \ell \le n$ and $E^{ij} \in M_{m\times n}(\mathbb{F})$ with

$$(E^{ij})_{k\ell} = \begin{cases} 1 & \text{if } i=k, j=\ell \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{E^{ij} \in M_{m\times n}(\mathbb{F}) : 1 \le i \le m, 1 \le j \le n\}$ is a basis of $M_{m\times n}(\mathbb{F})$.

(eg. If $m=2, n=3$, then $E^{21} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$)

eg Let $A = \begin{bmatrix} 1 & 2 & 2 & 3 \\ 2 & 4 & 3 & 4 \\ 1 & 2 & 1 & 1 \end{bmatrix}$. Find a basis for its null space $\operatorname{Null}(A)$

Sol

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 2 & 4 & 3 & 4 & 0 \\ 1 & 2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{cccc} \text{pivot} & \text{free} & \text{pivot} & \text{free} \end{array}$$

Let $x_2 = s$, $x_4 = t$

$x_3 + 2x_4 = 0 \Rightarrow x_3 = -2t$

$x_1 + 2x_2 + 2x_3 + 3x_4 = 0 \Rightarrow x_1 = -2s + t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2s+t \\ s \\ -2t \\ t \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Basis $= \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$

Rmk Vectors found in this way are lin. indept

Prop 2.21 Let $S = \{v_1, \cdots, v_n\} \subseteq V$. Then

① S is a basis of V $\Leftrightarrow$

② For any $v \in V$, $\exists$ unique $a_1, \cdots, a_n \in \mathbb{F}$ such that $v = a_1 v_1 + \cdots + a_n v_n$

Pf ($\Rightarrow$) Suppose S is a basis.

Then by defn, $V = \operatorname{span} S$

$\Rightarrow$ For any $v \in V$, $\exists$ $a_1, \cdots, a_n \in \mathbb{F}$ such that

$$v = a_1 v_1 + \cdots + a_n v_n$$

For uniqueness of a_i's, suppose

$$v = a_1 v_1 + \cdots + a_n v_n = b_1 v_1 + \cdots + b_n v_n$$

Then $\vec{0} = (a_1 - b_1) v_1 + \cdots + (a_n - b_n) v_n$

S is lin indept $\Rightarrow$ $a_i - b_i = 0$ $\forall i$

$\Rightarrow$ $a_i = b_i$ $\forall i$

$\therefore$ each a_i is unique $\Rightarrow$ ②

($\Leftarrow$) Suppose ② is true

Existence of a_i for any $v \in V$ $\Rightarrow$ $V = \operatorname{span} S$

To show S is lin. indept, suppose

$$\vec{0} = a_1 v_1 + \cdots + a_n v_n$$

Note $\vec{0} = 0 \cdot v_1 + \cdots + 0 \cdot v_n$

Uniqueness of a_i $\Rightarrow$ Each $a_i = 0$ $\Rightarrow$ S is lin. indept.

$\therefore$ S is a basis

Rmk In Prop 2.21, S is assumed to be finite

In general, for any subset $S \subseteq V$

S is a basis of V $\Leftrightarrow$

Every vector in V can be expressed as a linear combination of vectors from S in a unique way

Prop 2.31 Let $S = \{v_1, \cdots, v_n\}$ and $\operatorname{span} S = V$. Then $\exists\, S' \subseteq S$ such that S' is a basis of V

Pf If S is lin indept, take $S' = S$.

Otherwise, by Lemma 2.21, $\exists j$ s.t.

$$\operatorname{span}(S \setminus \{v_j\}) = \operatorname{span} S$$

If $S \setminus \{v_j\}$ is lin. indept, take $S' = S \setminus \{v_j\}$

Otherwise, continue the process and remove vectors from S until the remaining vectors are linearly independent.

Let S' be the resulting set. Then

$$\operatorname{span} S' = \operatorname{span} S = V$$

and S' is lin. indept by construction.

$\therefore$ $S' \subseteq S$ and is a basis of V

Thm 2.32 Every finite dimensional vector space has a basis (which is finite)

Pf By defn, a finite dim vector space V has a finite spanning set S (ie. $\operatorname{span} S = V$)

Prop 2.31 $\Rightarrow$ a subset of S is a basis of V

Rmk In Thm 2.32, the vector space is assumed to be finite dim. More generally

$$\boxed{\text{EVERY vector space has a basis}}$$

The proofs of this and generalizations of some other results in this section make use of axioms in set theory (eg. Zorn's lemma)

Prop 2.33 Let V be finite dimensional. Suppose $S \subseteq V$ is linear indept. Then $\exists$ a basis S' of V containing S

Pf Thm 2.32 $\Rightarrow$ V has a basis $\{v_1, \cdots, v_n\}$

By Prop 2.23, one can deduce $|S| \leq n$

Let $S = \{u_1, \cdots, u_m\}$. Apply lemma 2.21 successively to remove vectors from the list

$$u_1, \cdots, u_m, v_1, \cdots, v_n$$

to produce a lin. indept. set S' with same span

Note that no u_k was removed because

$$u_k \notin \operatorname{span}\{u_1, \cdots, u_{k-1}\}$$

Hence, by construction,

① $S \subset S'$

② $\operatorname{span} S' = \operatorname{span}\{u_1, \cdots, u_m, v_1, \cdots, v_n\} = V$

③ S' is lin. indept.

$\therefore$ S' is a basis of V containing S.

Prop 2.34 Let V be finite dimensional. $U \subseteq V$ be a subspace. Then $\exists$ a subspace $W \subseteq V$ such that $V = U \oplus W$

Pf $U \subseteq V \Rightarrow U$ is finite dim by Prop 2.26

Thm 2.32 $\Rightarrow$ U has a basis $S = \{u_1, \cdots, u_m\}$

S is lin. indept $\Rightarrow$

S can be extended to a basis of V

$$S' = \{u_1, \cdots, u_m, v_1, \cdots, v_n\}$$

Let $W = \operatorname{span}\{v_1, \cdots, v_n\}$

We want to show $V = U \oplus W$

Let $v \in V$. Then $\operatorname{span} S' = V \Rightarrow \exists a_i, b_j \in \mathbb{F}$ s.t.

$$v = a_1 u_1 + \cdots + a_m u_m + b_1 v_1 + \cdots + b_n v_n$$

Note $a_1 u_1 + \cdots + a_m u_m \in U$, $b_1 v_1 + \cdots + b_n v_n \in W$

$\Rightarrow$ $V \subseteq U + W$.

Clearly $U + W \subseteq V$. $\therefore$ $V = U + W$

To show $U+W$ is a direct sum,

Suppose $v \in U \cap W$.

Then $\exists\ a_i, b_j \in \mathbb{F}$ s.t.

$v = a_1u_1 + \cdots + a_mu_m = b_1v_1 + \cdots + b_nv_n$

$\Rightarrow \vec{0} = a_1u_1 + \cdots + a_mu_m + (-b_1)v_1 + \cdots + (-b_n)v_n$

S' is lin. indept $\Rightarrow a_i = -b_j = 0 \quad \forall i, j$

$\therefore\ v = 0u_1 + \cdots + 0u_m = \vec{0}$

$\therefore\ U \cap W \subseteq \{0\}$

Clearly $\{0\} \subseteq U \cap W \Rightarrow U \cap W = \{0\}$

$\therefore\ V = U \oplus W$