2017-18 MATH1010J
Lecture 17: Anti-deriviatives
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1 Antiderivatives and indefinite integrals

Definition 1.1. *Antiderivatives and Indefinite Integrals Let a function $f(x)$ be given. An* ***antiderivative*** *of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.*

The set of all antiderivatives of $f(x)$ is the ***indefinite integral of*** *f, denoted by*

$$\int f(x)\ dx.$$

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Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them. We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 1.1. *Antiderivative Forms Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that*

$$G(x) = F(x) + C.$$

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Proof. Suppose $F'(x) = G'(x) = f(x)$, then $\frac{d}{dx}(F(x) - G(x)) = 0$. So $F(x) - G(x)$ is a constant. □

Given a function f and one of its antiderivatives F, we know *all* antiderivatives of f have the form $F(x) + C$ for some constant C. Using Definition 1.1, we can say that

$$\int f(x)\ dx = F(x) + C.$$

Example 1.1. *Note that*

$$\frac{d}{dx}\left(\frac{x^2}{2}\right) = x.$$

(a) Find all antiderivatives of $f(x) = x$.

(b) Find the antiderivative of $f(x) = x$ that passes through the point $(0, 0)$.

Answer.

(a) Any derivative of $f(x)$ has the form

$$F(x) = \frac{x^2}{2} + C$$

where C is a real number.

(b) Because $F(0) = (0^2/2) + C = C$, Suppose $F(0) = 0$, then $C = 0$. $F(x) = \frac{x^2}{2}$.

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Definition 1.2. *The antiderivative is denoted by*

$$\int f(x)\,dx = F(x) + C,$$

where dx identifies x as the variable and C is a constant indicating that there are many possible antiderivatives, each varying by the addition of a constant. This is often called the **indefinite integral**.

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Remark. It is useful to remember that if you have performed an indefinite integration calculation that leads you to believe that $\int f(x)\,dx = G(x) + C$, then you can check your calculation by differentiating $G(x)$:

If $G'(x) = f(x)$, then the integration $\int f(x)\,dx = G(x) + C$ is correct, but if $G'(x)$ is anything other than $f(x)$, you've made a mistake.

The fact that indefinite integration and differentiation are reverse operations, except for the addition of the constant of integration, can be expressed symbolically as

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = f(x)$$

and

$$\int F'(x)\,dx = F(x) + C.$$

2 Basic integration formulas

The relationship between differentiation and antidifferentiation enables us to establish the following integration rules by "reversing" analogous differentiation rules.

Proposition 2.1 (The constant rule)**.**

$$\int k\,dx = kx + C \qquad \textit{for constant } k.$$

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Proof. Exercise. □

Proposition 2.2 (The power rule)**.**

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad \textit{for all } n \neq -1$$

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Proof. It is enough to show that the derivative of $\frac{x^{n+1}}{n+1}$ is x^n:

$$\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = \frac{1}{n+1}[(n+1)x^n] = x^n.$$

□

Proposition 2.3 (The logarithmic rule)**.**

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad \textit{for all } x \neq 0.$$

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Proof. If $x > 0$, then $|x| = x$ and

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

If $x < 0$, then $|x| = -x$ and

$$\frac{d}{dx}(\ln|x|) = \frac{d}{dx}[\ln(-x)] = \frac{1}{(-x)}(-1) = \frac{1}{x}.$$

Thus, for all $x \neq 0$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x},$$

so

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

□

Proposition 2.4 (The exponential rule)**.**

$$\int e^{kx}\, dx = \frac{1}{k}e^{kx} + C \qquad \textit{for constant } k \neq 0.$$

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Proof. Exercise. □

Example 2.1. *Compute*

$$\int 3x^7 dx.$$

Answer.

$$\begin{aligned}\int 3x^7 dx &= 3\int x^7 dx \\ &= 3 \cdot \frac{x^8}{8} + C.\end{aligned}$$

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Example 2.2. *Compute*

$$\int \frac{1}{\sqrt{x}}\, dx.$$

Answer.

$$\int \frac{1}{\sqrt{x}}\,dx = \int x^{-1/2}\,dx = \frac{1}{1/2}x^{1/2} + C = 2\sqrt{x} + C$$

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Example 2.3. *Compute*

$$\int e^{-3x}\,dx.$$

Answer.

$$\int e^{-3x}\,dx = \frac{1}{-3}e^{-3x} + C.$$

■

Proposition 2.5 (Algebraic rules for indefinite integration)**.**

- *The* **constant multiple rule**

$$\int kf(x)\,dx = k\int f(x)\,dx \qquad \textit{for constant } k.$$

- *The* **sum rule**

$$\int [f(x) + g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

- *The* **difference rule**

$$\int [f(x) - g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx.$$

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Example 2.4. *Show that*

$$\int \cos x = \sin x + C.$$

Answer.

$$\frac{d}{dx}\sin x = \cos x.$$

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Proposition 2.6 (Algebraic rules for indefinite integration)**.**

- *The* **constant multiple rule**

$$\int kf(x)\,dx = k\int f(x)\,dx \qquad \text{for constant } k.$$

- *The* **sum rule**

$$\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx.$$

- *The* **difference rule**

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx.$$

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In below we use a table to summarize the main results.

$\frac{d}{dx}\big(cf(x)\big) = c\cdot f'(x)$	$\int c\cdot f(x)\ dx = c\cdot \int f(x)\ dx$
$\frac{d}{dx}\big(f(x)\pm g(x)\big) = f'(x)\pm g'(x)$	$\int \big(f(x)\pm g(x)\big)\ dx = \int f(x)\ dx \pm \int g(x)\ dx$
$\frac{d}{dx}\big(C\big) = 0$	$\int 0\ dx = C$
$\frac{d}{dx}\big(x\big) = 1$	$\int 1\ dx = \int dx = x + C$
$\frac{d}{dx}\big(x^n\big) = n\cdot x^{n-1}$	$\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C \quad (n\neq -1)$
$\frac{d}{dx}\big(\sin x\big) = \cos x$	$\int \cos x\ dx = \sin x + C$
$\frac{d}{dx}\big(\cos x\big) = -\sin x$	$\int \sin x\ dx = -\cos x + C$
$\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	$\int \sec^2 x\ dx = \tan x + C$
$\frac{d}{dx}\big(\csc x\big) = -\csc x\cot x$	$\int \csc x\cot x\ dx = -\csc x + C$
$\frac{d}{dx}\big(\sec x\big) = \sec x\tan x$	$\int \sec x\tan x\ dx = \sec x + C$
$\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$	$\int \csc^2 x\ dx = -\cot x + C$
$\frac{d}{dx}\big(e^x\big) = e^x$	$\int e^x\ dx = e^x + C$
$\frac{d}{dx}\big(a^x\big) = \ln a\cdot a^x$	$\int a^x\ dx = \frac{1}{\ln a}\cdot a^x + C$
$\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$	$\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$

Example 2.5. *Find the following integrals:*

(a) $\displaystyle\int (2x^5 + 8x^3 - 3x^2 + 5)\,dx.$

(b) $\displaystyle\int \left(\frac{x^3+2x-7}{x}\right)\,dx.$

(c) $\int (3e^{-5t} + \sqrt{t})\, dt.$

Answer.

(a)

$$\begin{aligned} \int (2x^5 + 8x^3 - 3x^2 + 5)\, dx &= 2\int x^5\, dx + 8\int x^3\, dx - 3\int x^2\, dx + \int 5\, dx \\ &= 2\left(\frac{x^6}{6}\right) + 8\left(\frac{x^4}{4}\right) - 3\left(\frac{x^3}{3}\right) + 5x + C \\ &= \frac{1}{3}x^6 + 2x^4 - x^3 + 5x + C. \end{aligned} \tag{1}$$

(b)

$$\begin{aligned} \int \left(\frac{x^3 + 2x - 7}{x}\right) dx &= \int \left(x^2 + 2 - \frac{7}{x}\right) dx \\ &= \frac{1}{3}x^3 + 2x - 7\ln|x| + C. \end{aligned} \tag{2}$$

(c)

$$\begin{aligned} \int (3e^{-5t} + \sqrt{t})\, dt &= \int (3e^{-5t} + t^{1/2})\, dt \\ &= 3\left(\frac{1}{-5}e^{-5t}\right) + \frac{1}{3/2}t^{3/2} + C \\ &= -\frac{3}{5}e^{-5t} + \frac{2}{3}t^{3/2} + C. \end{aligned} \tag{3}$$

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Example 2.6. *Find the function $f(x)$ whose tangent has slope $4x^3 + 5$ for each value of x and whose graph passes through the point (1, 10).*

Answer. The slope of the tangent at each point $(x, f(x))$ is the derivative $f'(x)$. Thus,

$$f'(x) = 4x^3 + 5$$

and so $f(x)$ is the antiderivative

$$\int f'(x)\,dx = \int (4x^3+5)\,dx = x^4 + 5x + C.$$

To find C, use the fact that the graph of f passes through $(1, 10)$. That is, substitute $x = 10$ and $f(1) = 10$ into the equation for $f(x)$ and solve for C to get

$$10 = (1)^4 + 5(1) + C \qquad \text{or} \qquad C = 4.$$

Thus, the desired function is $f(x) = x^4 + 5x + 4$. ■

3 Anti-derivatives of piecewise continuous function

Example 3.1. *Let*

$$f(x) = \begin{cases} x^2 - 1 & x < 1, \\ x - 1 & x \geq 1. \end{cases}$$

Answer. We can defined

$$F(x) = \begin{cases} \int (x^2-1)dx = \frac{x^3}{3} - x + C_1 & x < 1 \\ \int (x-1)dx = \frac{x^2}{2} - x + C_2 & x \geq 1, \end{cases}$$

where C_1 and C_2 are constants to be determined.
In order for the funciton to be continuous

$$\lim_{x \to 1^-} \frac{x^3}{3} - x + C_1 = \lim_{x \to 1^+} \frac{x^2}{2} - x + C_2.$$

Therefore

$$C_1 - \frac{2}{3} = C_2 - \frac{1}{2}.$$

We then express C_2 in terms of C_1:

$$C_2 = C_1 - \frac{1}{6}.$$

Write $C_1 = C$.

$$F(x) = \begin{cases} \frac{x^3}{3} - x + C, & x < 1, \\ \frac{x^2}{2} - x - \frac{1}{6} + C, & x \geq 1, \end{cases}$$

where C is a constant. ■

Example 3.2. *Let*

$$f(x) = \begin{cases} x^2 - 1 & x < 1, \\ x - 1 & 1 \leq x \geq 2, \\ e^{x-2} & x > 2. \end{cases}$$

Answer.

$$F(x) = \begin{cases} \int (x^2 - 1)dx = \frac{x^3}{3} - x + C_1 & x < 1, \\ \int (x - 1)dx = \frac{x^2}{2} - x + C_2 & 1 \leq x \leq 2, \\ \int e^{x-2}dx = e^{x-2} + C_3 & x > 2. \end{cases}$$

$$\lim_{x \to 1^-} \frac{x^3}{3} - x + C_1 = \lim_{x \to 1^+} \frac{x^2}{2} - x + C_2.$$

Hence

$$C_2 = C_1 - \frac{1}{6}.$$

Also

$$\lim_{x \to 2^-} \frac{x^2}{2} - x + C_2 = \lim_{x \to 2^+} e^{x-2} + C_3.$$

Hence

$$C_3 = C_2 + 2 - 2 - 1 = C_2 - 1 = C_1 - \frac{7}{6}.$$

Thus

$$F(x) = \begin{cases} \frac{x^3}{3} - x + C & x < 1, \\ \frac{x^2}{2} - x + C - \frac{1}{6} & 1 \leq x \leq 2, \\ e^{x-2} + C - \frac{7}{6} & x > 2, \end{cases}$$

where C is a constant. ■