香港中文大學
學生會

Week 2

Monotone Convergence Theorem (Revisited)

If $\{a_n\}$ is $\begin{cases} \text{nondecreasing and bounded above,} \\ \text{nonincreasing and bounded below,} \end{cases}$ then $\{a_n\}$ converges.

Example

- $a_1 = 2$
- $a_{n+1} = -\frac{1}{a_n} + 2, \ n \geq 1$

Last time, we proved

Step 1. $a_n \geq 1$
Step $\_$. $a_{n+1} \leq a_n$

$\Longrightarrow$ $\{a_n\}$ converges by Monotone convergence thm.

Q. $\lim_{n\to\infty} a_n = \quad ?$

A. Let $x = \lim_{n\to\infty} a_n$. Then $x \geq 1$ because $a_n \geq 1$ for all $n \in \mathbb{N}$ by Step 1.

- $\lim_{n\to\infty} a_{n+1} = x$
- $\lim_{n\to\infty} \left(-\frac{1}{a_n} + 2\right) = -\frac{1}{x} + 2$

$\Rightarrow$ $x = -\frac{1}{x} + 2 \Longrightarrow \frac{1}{x}(x^2 - 2x + 1)$

$= \frac{1}{x}(x-1)^2 = 0$

$\therefore x = 1$. ▨

Exercise $\begin{cases} a_1 = 1 \\ a_{n+1} = \frac{1}{a_n} + 1 \end{cases}$

Show that the sequence $\{a_n\}$ converges to $\frac{1+\sqrt{5}}{2}$.

Hint: Consider $\begin{cases} b_n = a_{2n} \\ c_n = a_{2n-1} \end{cases}$, $\forall n \in \mathbb{N}$.

Step 1. $b_n \geq \frac{1+\sqrt{5}}{2}$ and $c_n \leq \frac{1+\sqrt{5}}{2}$ $\forall n \in \mathbb{N}$

Step 2. $b_{n+1} \leq b_n$ and $c_{n+1} \geq c_n$ $\forall n \in \mathbb{N}$

($\Rightarrow$ Both $\{b_n\}$ and $\{c_n\}$ converge.)

Step 3. $\lim_{n\to\infty} b_n = \frac{1+\sqrt{5}}{2} = \lim_{n\to\infty} c_n$

Step 4. $\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}$. ▨

Function

A function $f: A \to B$ is a rule of correspondence from one set A to another set B.

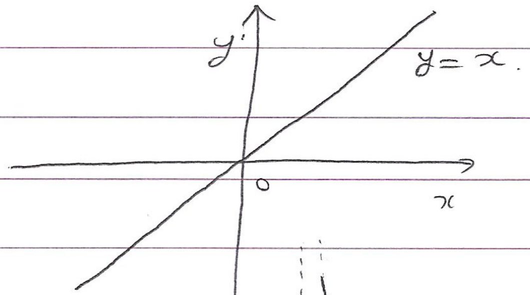
Here, $\begin{cases} A \text{ is called the } \underline{\text{domain}} \text{ of } f. \\ B \text{ is called the } \underline{\text{codomain}} \text{ of } f. \end{cases}$

For $a \in A$, $f(a) \in B$ is called the value of f at a.

Examples

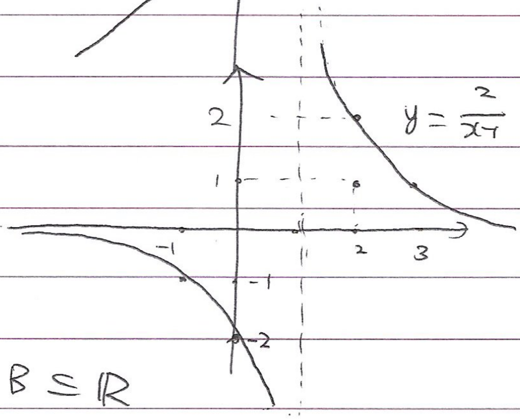
① $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x$



② $g: \mathbb{R} \setminus \{1\} \to \mathbb{R}$

$g(x) = \frac{2}{x-1}$

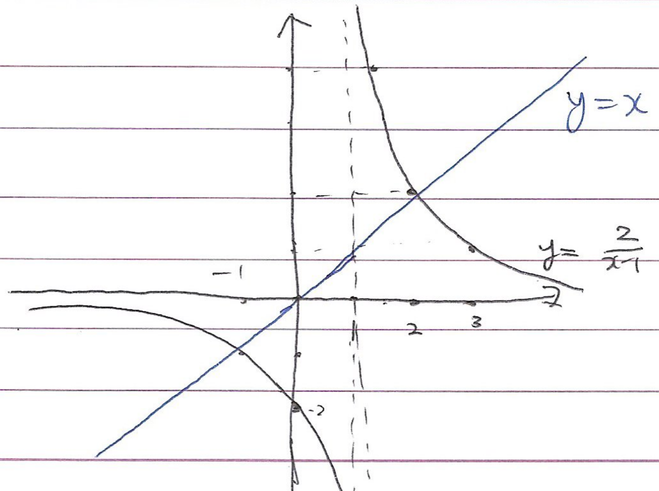


Graph of function $A, B \subseteq \mathbb{R}$

For a function $f: A \to B$, the graph of f is the set of all points (x, y) in the xy-plane where $x \in A$ and $y = f(x)$.

Exercise

Find all values of x for which $x > \frac{2}{x-1}$.



Answer: $-1 < x < 1$ and $x > 2$.

Def $f, g : A \to \mathbb{R}$ functions

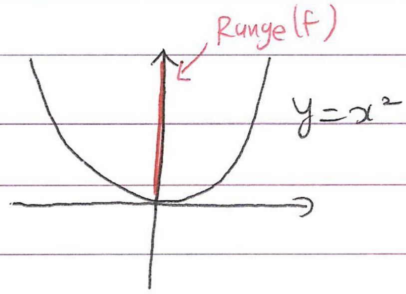
- Sum / difference of f, g is a function
  $f \pm g : A \to \mathbb{R}$ defined by
  $$\begin{cases} (f+g)(a) = f(a) + g(a) & \forall a \in A \\ (f-g)(a) = f(a) - g(a) \end{cases}$$
- Product of f, g is a function
  $fg : A \to \mathbb{R}$ defined by
  $$(fg)(a) = f(a)\, g(a) \quad \forall a \in A.$$
- Quotient ~~of~~ $\frac{f}{g}$ is a function
  $\frac{f}{g} : A' \to \mathbb{R}$ defined by
  $$\left(\frac{f}{g}\right)(a) = \frac{f(a)}{g(a)} \quad \forall a \in A'$$
  where $A' = \{a \in A \mid g(a) \neq 0\}$.

- Composition of two functions $f : A \to B$ and $g : B \to C$ is a function
  $g \circ f : A \to C$ defined by
  $$(g \circ f)(a) = g(f(a)) \quad \forall a \in A$$

Def • Range or Image of $f : A \to B$ is the set
$$\text{Range}(f) = \{b \in B \mid b = f(a) \text{ for some } a \in A\}$$
- If $\text{Range}(f) = B$, then f is surjective or onto.
- If $f(a) \neq f(a')$ for all $a \neq a'$, then f is injective or one-to-one.

Examples

① $f : \mathbb{R} \to \mathbb{R}$
$f(x) = x^2$.

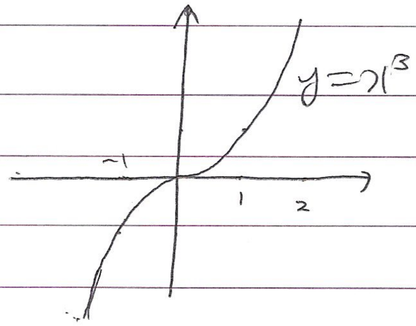


- $\text{Range}(f) = \{y \in \mathbb{R} \mid y \geq 0\} = \mathbb{R}_{\geq 0}$
  $\because \forall y \geq 0, \quad f(\sqrt{y}) = (\sqrt{y})^2 = y.$
- $\therefore$ f is not surjective because $\text{Range}(f) \neq \mathbb{R}$.

- f is not injective because $f(1) = 1 = f(-1)$.

② $f: \mathbb{R} \to \mathbb{R}$

$f(x) = x^3$



- Range $(f) = \mathbb{R}$

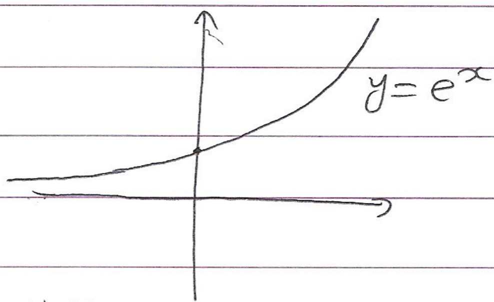
$\because \forall y \in \mathbb{R}$, $f(\sqrt[3]{y}) = y$.

- Hence f is surjective
- Also, f is injective.

$\because$ For all $a \neq a' \in \mathbb{R}$, we may assume that $a < a'$ without loss of generality. Since the function f is strictly increasing, $f(a) < f(a')$ and hence $f(a) \neq f(a')$. ▯

③ $f: \mathbb{R} \to \mathbb{R}$

$f(x) = e^x$



- Range (f)

$= \{y \in \mathbb{R} \mid y > 0\} = \mathbb{R}_{>0}$

- Hence f is not surjective.
- But f is injective since f is strictly increasing as in ②.

Def Suppose $f: A \to B$ is an injective function.

Then the inverse function of f is a function

$f^{-1}: \text{Range}(f) \to A$ defined by

$f^{-1}(b) = a$ where $a \in A$ satisfies $f(a) = b$.

Note that the inverse function is well-defined because f is injective.

Examples

① $f: \mathbb{R} \to \mathbb{R}$ is injective and $\text{Range}(f) = \mathbb{R}$.
$f(x) = x^3$

$\Rightarrow f^{-1}: \mathbb{R} \to \mathbb{R}$ is a function defined by
$f(y) = \sqrt[3]{y}$.

② $f: \mathbb{R} \to \mathbb{R}$ is injective and $\text{Range}(f) = \mathbb{R}_{>0}$
$f(x) = e^x$

$\Rightarrow f^{-1}: \mathbb{R}_{>0} \to \mathbb{R}$ is a function defined by
$f(y) = \ln y \, (= \log_e y)$.

Limits of functions

Let $f: A \to \mathbb{R}$ be a function.

Def We say that the limit of f at $a \in A$ is L if
$\forall \varepsilon > 0$, $\exists \delta > 0$ such that
$|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.
In this case, we use the notation $\lim\limits_{x \to a} f(x) = L$.

Examples

① $f(x) = x^2$

$\Rightarrow \lim\limits_{x \to 0} f(x) = 0$.

$\because$ Given $\varepsilon > 0$, let $\delta = \sqrt{\varepsilon}$.
Then we have $|f(x) - 0| = |x^2| < \varepsilon$ whenever $0 < |x| < \delta = \sqrt{\varepsilon}$.