MATH1010E (Wk1.1,1.2)

Keywords: Brief discussion about the Prerequisites, (Monday); Function, Domain, Codomain, Range, Sequence, How to find $\sqrt{2}$ using sequences.

Introduction.

In the following, we will start from Wednesday's lecture. As for the materials covered on Monday, please test yourself via the E-exercises (whose link I will send to you later this evening if I have time).

Some Simple Functions from School Math

(Polynomial Function) First we have the polynomial functions, they are objects written in the form $a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n$

(This kind of expressions is called "degree *n* polynomial, provided $a_n \neq 0$).

Concrete Examples: (degree 1 polynomial) $a + bx$, where $b \neq 0$

(degree 2 polynomial) $a + bx + cx^2$, where $c \neq 0$

(degree 3 polynomial) $a + bx + cx^2 + dx^3$, where $d \neq 0$

(degree 4 polynomial) $a + bx + cx^2 + dx^3 + ex^4$, where $e \neq 0$

We also have the

Rational functions

These are functions of the form $\frac{polynomial}{polynomial}$.

Example

$$\frac{1 + 3x + x^4}{2 - 3x + x^3}$$

Furthermore we have the

Trigonometric Functions

By this we mean (i) trigonometric functions like $\sin(x)$, $\cos(x)$, $\tan(x)$, or $\sec(x)$, $\csc(x)$, $\cot(x)$

Propertie(s): Roughly they are functions which <u>cannot be written</u> as polynomials!

Remark:

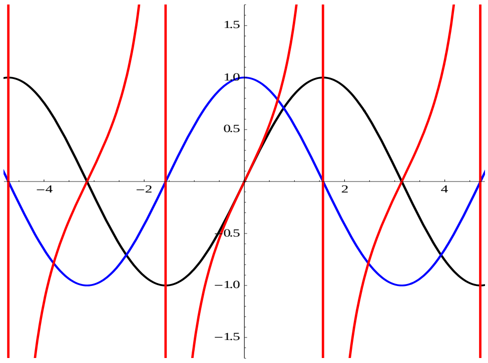
Make sure that you know the picture of each of the trigonometric functions.

Pictures of Simple Trigo. Functions

$$\sin(x), \cos(x), \tan(x)$$

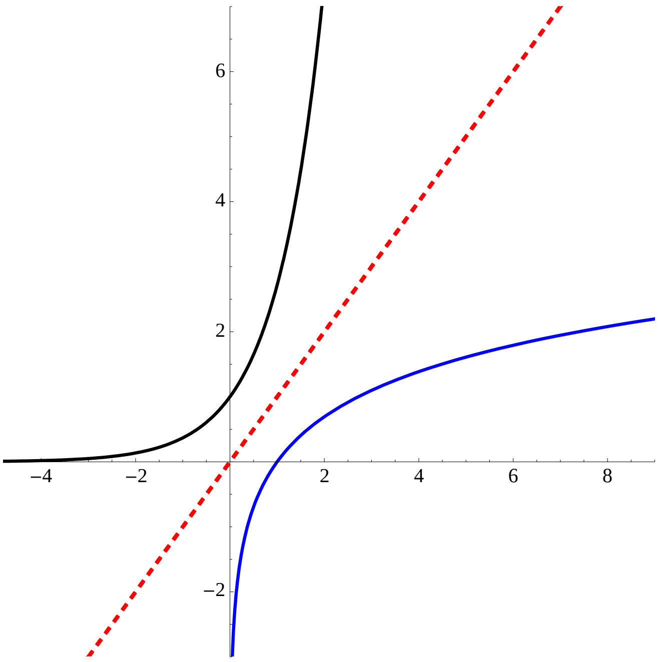
Question: In the following picture, can you recognize which is which?

Question: In the following picture, there are some computer errors producing lines which cannot exist in reality. Which lines are they?



Finally, we have the exponential function, i.e. $\exp(x)$ and the logarithm function, i.e. $\ln(x)$.

Their pictures are (Question: Which is which?)



Abstract Definition of Function

Now let us give a "rather abstract" definition of function, which you may not have learned in school math. In some school math books, each individual function is usually defined each time by a single-line formula e.g. $f(x) = x^2 + 3x - 2$ or $g(x) = \frac{\sin x}{x^2+3}$..

One difficulty such definition may lead to is that it cannot handle more complicated functions, such as

$$\text{abs(x)} = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

which is the "absolute value" function which is defined by a three-line formula.

Remark

Traditionally, this function is denoted by $|x|$.

Point

What the "absolute value function" tells us is that sometimes a function cannot be defined by a one-line formula. This leads to a more modern and "abstract" definition of function, outlined below:

Definition of Function

A function is a <u>rule</u>, say f, assigning a <u>unique</u> value, denoted by $f(x)$, to any given value x ($f(x)$ – "value" or "evaluation" of f at x).

Definition of Domain. Definition of Range

The collection of all such x is called Domain of f. (Notation: Dom(f) or Domain(f))

The collection of all such $f(x)$ is called the Range of f. (Notation: R(f) or Range(f)).

A Useful & Convenient Concept – Codomain

Very often, we want to be able to talk about "where" the values $f(x)$ of the function f lives in. A lazy answer is the set of all real numbers $\mathbb{R}$. Why is it a "lazy answer"? It is because anyway, $f(x)$ must be a real number.

Point

What we really want to know is the range, but before finding it, we need a "big enough set" to contain all possible values like $f(x)$. This "big enough set" is the codomain.

Two Examples Plus One Question

1. Let $\mathbb{R}$ be the domain of the function $f(x) = \sin(x)$. Let its codomain be $\mathbb{R}$. Then from the picture (it is NOT A PROOF!) we see that the range is the closed interval $[-1,1]$.

2. (This example can be proved rigorously) Consider the function $\mathbb{R}\backslash\{1\} \xrightarrow{f} \mathbb{R}$ given by $f(x) = \frac{x}{x-1}$. Find $R(f)$

   Suggested Solution. Recall from the definition of range that

   $R(f) = \{y \in \text{Codomain of } f \mid y = f(x) \text{ for some } x \in \text{Dom}(f)\}$

   So the main issue is to find "for which y" the equation $y = f(x)$ has a solution x in $\text{Dom}(f)$.

   To find all such y (plural!), we study the equation $y = \frac{x}{x-1}$. After simplification, we obtain $y(x-1) = x \Rightarrow x - xy = -y \Rightarrow x = \frac{-y}{1-y} = \frac{y}{y-1}$

   From this formula, we see that x can be obtained, whenever $y \neq 1$, because then the right-hand side, i.e. $\frac{y}{y-1}$ is computable.

   Conclusion: The range of this function is the set

   $$\{y \in \mathbb{R} \mid y \neq 1\}$$

3. **Question for You**

   Consider now the function $\mathbb{R}\backslash\{9\} \xrightarrow{f} \mathbb{R}$ given by $f(x) = \frac{x^2+9}{x-9}$. Find the range of f.

Additional Short Questions

1. Let f be a function assigning (i.e. "giving") to each month of the year the initial alphabet of that month.
   - a What is $\text{Dom}(f)$?
   - b What is $\text{R}(f)$?
2. Let f be a function from the domain $\mathbb{R}$ to the codomain $\mathbb{R}$ defined by the rule: $f(x) = \frac{x}{x^2+1}$. Find (i) $f(-1)$, (ii) $f\left(f\left(f(-1)\right)\right)$.