THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 University Mathematics 2017-2018

Suggested Solution to Assignment 6

1. (a) $\int \sin^2 2x \sin 5x\,dx = \int \frac{1}{2}(1-\cos 4x)\sin 5x\,dx = \int \frac{1}{4}(-\sin x+2\sin 5x-\sin 9x)\,dx = \frac{1}{4}\cos x - \frac{1}{10}\cos 5x + \frac{1}{36}\cos 9x + C$

(b) $\int \cos^2 2x \sin^3 2x\,dx = -\frac{1}{2}\int \cos^2 2x \sin^2 2x\,d(\cos 2x) = -\frac{1}{2}\int \cos^2 2x(1-\cos^2 2x)\,d(\cos 2x) = -\frac{1}{6}\cos^3 2x + \frac{1}{10}\cos^5 2x + C$

(c) Let $y = x-2$,
$\int \frac{x-2}{\sqrt{x^2-4x+3}}\,dx = \int \frac{y}{\sqrt{y^2-1}}\,dy$
Let $y = \sec u$, then $dy = \sec u \tan u du$,
$\int \frac{y}{\sqrt{y^2-1}}\,dy = \int \frac{\sec u}{\sqrt{\sec^2 u - 1}}\sec u\tan u du = \int \frac{\sec u}{\tan u}\sec u \tan u du = \int \sec^2 u du = \tan u + C = \sqrt{y^2-1} + C = \sqrt{x^2-4x+3} + C$

(d) $\int \frac{e^{x-1}}{1+e^{2x}}\,dx = \frac{1}{e}\int \frac{e^x}{1+e^{2x}}\,dx = \frac{1}{e}\int \frac{1}{1+(e^x)^2}\,d(e^x) = \frac{1}{e}\tan^{-1}(e^x) + C$

(e) $\int x\sin^{-1}x\,dx = \int \sin^{-1}x\,d(\frac{x^2}{2}) = \frac{x^2\sin^{-1}x}{2} - \int \frac{x^2}{2}\,d(\sin^{-1}x) = \frac{x^2\sin^{-1}x}{2} + \int \frac{x^2}{2\sqrt{1-x^2}}\,dx$
Let $x = \sin u$, then $dx = \cos u\,du$.
$\int \frac{x^2}{2\sqrt{1-x^2}}\,dx = \int \frac{1}{2}\sin^2 u\,du = \int \frac{1}{4}(1-\cos 2u)\,du = \frac{u}{4} - \frac{\sin 2u}{8} + C = \frac{\sin^{-1}x}{4} - \frac{x\sqrt{1-x^2}}{4} + C$
Therefore, $\int \frac{x^2}{2\sqrt{1-x^2}}\,dx = \frac{x^2\sin^{-1}x}{2} + \frac{\sin^{-1}x}{4} - \frac{x\sqrt{1-x^2}}{4} + C$

(f)

$$\begin{aligned}
\int \cos(\ln x)\,dx &= x\cos(\ln x) - \int x\,d(\cos(\ln x)) \\
&= x\cos(\ln x) + \int \sin(\ln x)\,dx \\
&= x\cos(\ln x) + x\sin(\ln x) - \int x\,d(\sin(\ln x)) \\
&= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\,dx \\
\therefore \quad \int \cos(\ln x)\,dx &= \frac{x}{2}(\cos(\ln x) + \sin(\ln x)) + C
\end{aligned}$$

2. (a) $\int_4^6 |2x-1|\,dx = \int_4^6 2x-1\,dx = [x^2-x]_4^6 = 18$

(b) $\int_0^{2\pi} |1+2\cos x|\,dx = \int_0^{2\pi/3} 1+2\cos x\,dx + \int_{2\pi/3}^{4\pi/3} -(1+2\cos x)\,dx + \int_{4\pi/3}^{2\pi} 1+2\cos x\,dx = 4\sqrt{3} + \frac{2\pi}{3}$

(c) $\int_1^e x^2 \ln x\,dx = \int_1^e \ln x\, d(\frac{x^3}{3}) = \left[\frac{x^3 \ln x}{3}\right]_1^e - \int_1^e \frac{x^2}{3}\,dx = \frac{2e^3+1}{9}$

3. (a) $\lim_{n\to\infty} \frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n(n+1)}} + \cdots + \frac{1}{\sqrt{n(2n-1)}} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{n(n+k)}} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{1}{\sqrt{1+\frac{k}{n}}} \cdot \frac{1}{n} =$ $\int_0^1 \frac{1}{\sqrt{1+x}}\,dx = 2(\sqrt{2}-1)$

(b) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{k^2+n^2} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{(\frac{k}{n})^2+1} \cdot \frac{1}{n} = \int_0^1 \frac{1}{x^2+1}\,dx = \frac{\pi}{4}$

4. (a) $\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^{\sin x} \sin(e^t)\,dt - \int_0^x \sin(e^t)\,dt\right) = \sin(e^{\sin x})\cos x - \sin(e^x)$

(b) $\frac{dy}{dx} = \frac{d}{dx}\left(\int_0^x \sin(e^x+e^t)\,dt\right) = \frac{d}{dx}\left(\int_0^x \sin(e^x)\cos(e^t) + \cos(e^x)\sin(e^t)\,dt\right)$

$= \frac{d}{dx}\left(\sin(e^x)\int_0^x \cos(e^t)\,dt + \cos(e^x)\int_0^x \sin(e^t)\,dt\right)$

$= e^x \cos x \int_0^x \cos(e^t)\,dt + 2\sin(e^x)\cos(e^x) - e^x \sin x \int_0^x \sin(e^t)\,dt$

(c) Let $u = xt$, then $du = x\,dt$. When $t = 1$, $u = x$; $t = x$, $u = x^2$.

$y = \int_1^x \frac{e^{xt}}{t^2}\,dt = x\int_x^{x^2} \frac{e^u}{u^2}\,du$

$\frac{dy}{dx} = \frac{d}{dx}\left(x\int_x^{x^2} \frac{e^u}{u^2}\,du\right) = \int_x^{x^2} \frac{e^u}{u^2}\,du + x\frac{d}{dx}\left(\int_x^{x^2} \frac{e^u}{u^2}\,du\right)$

$= \int_x^{x^2} \frac{e^u}{u^2}\,du + x\frac{d}{dx}\left(\int_0^{x^2} \frac{e^u}{u^2}\,du - \int_0^x \frac{e^u}{u^2}\,du\right) = \int_x^{x^2} \frac{e^u}{u^2}\,du + x\left(\frac{e^{(x^2)}}{x^4} - \frac{e^x}{x^2}\right)$

$= \int_x^{x^2} \frac{e^u}{u^2}\,du + \frac{e^{(x^2)}}{x^3} - \frac{e^x}{x}$

5. Let $u = a - x$, then $-du = dx$. When $x = 0$, $u = a$; $x = a$, $u = 0$.

$$\begin{aligned}
\int_0^a f(a-x)\,dx &= -\int_a^0 f(u)\,du \\
&= \int_0^a f(u)\,du \\
&= \int_0^a f(x)\,dx \qquad \text{(dummy variable)}
\end{aligned}$$

By putting $a = \pi/2$ and $f(x) = \frac{\cos^3 x}{\sin x + \cos x}$, we have

$$\begin{aligned}
\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\,dx &= \int_0^{\pi/2} \frac{\cos^3(\pi/2 - x)}{\sin(\pi/2 - x) + \cos(\pi/2 - x)}\,dx \\
&= \int_0^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x}\,dx \\
\therefore \int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x}\,dx &= \frac{1}{2}\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x} + \frac{\sin^3 x}{\sin x + \cos x}\,dx \\
&= \frac{1}{2}\int_0^{\pi/2} 1 - \frac{1}{2}\sin 2x\,dx \\
&= \frac{\pi - 1}{4}
\end{aligned}$$

6. (a) Let $u = a - x$, then $-du = dx$. When $x = 0$, $u = a$; $x = a$, $u = 0$.

$$\begin{aligned}
\int_0^a f(x)g(x)\,dx &= -\int_a^0 f(a-u)g(a-u)\,du \\
&= \int_0^a f(u)(M - g(u))\,du \\
&= \int_0^a f(x)(M - g(x))\,dx \qquad \text{(dummy variable)} \\
&= M\int_0^a f(x)\,dx - \int_0^a f(x)g(x)\,dx \\
2\int_0^a f(x)g(x)\,dx &= M\int_0^a f(x)\,dx \\
\int_0^a f(x)g(x)\,dx &= \frac{M}{2}\int_0^a f(x)\,dx
\end{aligned}$$

(b) Put $a = \pi$, let $f(x) = \cos^2 x \sin^4 x$ and $g(x) = x$. Then, we have $f(x) = f(\pi - x)$ and $g(x) + g(\pi - x) = \pi$. By (a), we have

$$\begin{aligned}
\int_0^\pi x\cos^2 x\sin^4 x\,dx &= \frac{\pi}{2}\int_0^\pi \cos^2 x\sin^4 x\,dx \\
&= \frac{\pi}{2}\int_0^\pi \cos^2 x\sin^4 x\,dx \\
&= \frac{\pi}{64}\int_0^\pi 2 - \cos 2x - 2\cos 4x + \cos 6x\,dx \\
&= \frac{\pi^2}{32}
\end{aligned}$$

7. Note that for all $0 \le t \le x$, we have $e^t \le \sqrt{e^{2t}+1} \le \sqrt{e^{2t}+e^{2t}} = \sqrt{2}e^t$. Therefore

$$e^x - 1 = \int_0^x e^t\,dt \le \int_0^x \sqrt{e^{2t}+1}\,dx \le \int_0^x \sqrt{2}e^t\,dt = \sqrt{2}(e^x - 1).$$

8. (a) For $a < x < b$, we have

$$\begin{aligned}
F(x) &= \left(\int_a^x [f(t)]^2\,dt\right)\left(\int_a^x [g(t)]^2\,dt\right) - \left(\int_a^x f(t)g(t)\,dt\right)^2 \\
F'(x) &= [f(x)]^2\left(\int_a^x [g(t)]^2\,dt\right) + [g(x)]^2\left(\int_a^x [f(t)]^2\,dt\right) - 2f(x)g(x)\left(\int_a^x f(t)g(t)\,dt\right) \\
&= \left(\int_a^x [f(x)]^2[g(t)]^2\,dt\right) + \left(\int_a^x [f(t)]^2[g(x)]^2\,dt\right) - \left(\int_a^x 2f(x)f(t)g(x)g(t)\,dt\right) \\
&= \int_a^x [f(x)g(t)]^2 - 2f(x)f(t)g(x)g(t) + [f(t)g(x)]^2\,dt \\
&= \int_a^x \left((f(x)g(t) - f(t)g(x)\right)^2\,dt \\
&\ge 0
\end{aligned}$$

Also, $F(x)$ is continuous at $x = a$ and $x = b$, so $F(x)$ is increasing on $[a, b]$.

Therefore,

$$\begin{aligned}
F(b) &\geq F(a) \\
\left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right) - \left(\int_a^b f(x)g(x)\,dx\right)^2 &\geq 0 \\
\left(\int_a^b [f(x)]^2\,dx\right)\left(\int_a^b [g(x)]^2\,dx\right) &\geq \left(\int_a^b f(x)g(x)\,dx\right)^2
\end{aligned}$$

(b) Putting $a = q$, $b = p$, $f(x) = \dfrac{1}{x}$ and $g(x) = 1$, we have

$$\begin{aligned}
\left(\int_p^q \frac{1}{x^2}\,dx\right)\left(\int_p^q 1\,dx\right) &\geq \left(\int_p^q \frac{1}{x}\,dx\right)^2 \\
\left(\frac{1}{p} - \frac{1}{q}\right)(q-p) &\geq \left(\ln\left(\frac{p}{q}\right)\right)^2 \\
\frac{(q-p)^2}{pq} &\geq \left(\ln\left(\frac{p}{q}\right)\right)^2 \\
\ln\left(\frac{p}{q}\right) &\leq \frac{p-q}{\sqrt{pq}}
\end{aligned}$$