**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH1010 University Mathematics (Spring 2018)**
**Tutorial 5**
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You can apply mean value theorem or other results covered in MATH1010.
Those questions with $*$ may be challenging.

**Exercise 1**:

Show the following results.

(a) For $x \in [0, 1)$,
$$\log(1-x) \leq -x$$

(b) For $x \in \left[0, \frac{1}{2}\right]$,
$$-x - x^2 \leq \log(1-x)$$

(c) Let $c \in [0,1]$. For $x \in [0,1]$,
$$(1-c)^x \leq 1 - cx$$

Remark: For $x \in \left[0, \frac{1}{2}\right]$, by (a) and (b), we have
$$-x - x^2 \leq \log(1-x) \leq -x$$

**Exercise 2**:

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Suppose $f'$ is strictly increasing.

Show that

(a) For any $x \in \mathbb{R}$,
$$f'(x) < f(x+1) - f(x) < f'(x+1)$$

(b) For any $n \in \mathbb{N} \setminus \{1\}$,
$$f'(1) + f'(2) + ... + f'(n-1) < f(n) - f(1) < f'(2) + f'(3) + ... + f'(n)$$

**Exercise 3(***)**:

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

Let $\{x_n\} \subset \mathbb{R}$ be a sequence defined by
$$x_{n+1} = f(x_n)$$

Suppose there exists $M < 1$ such that $|f'(x)| \leq M$ for any $x \in \mathbb{R}$. Show that

(1) (**) There exists $z \in \mathbb{R}$ such that $f(z) = z$;

(2) There is only one $z \in \mathbb{R}$ that satisfies the equation $f(x) = x$;

(3) (*) $\lim_{n\to\infty} x_n = z$.

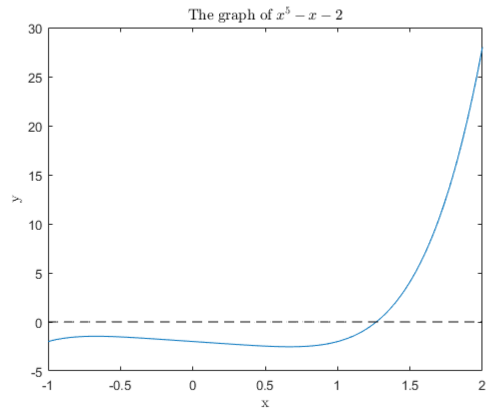
Remark: You may just attempt (3) by assuming (1), (2).

**Appendix**

In exercise 3, you are asked to show that the sequence $\{x_n\}$ converges to the fixed point $z$.

One application of this result is to find the roots of functions. For instance, consider the polynomial equation

$$x^5 - x - 2 = 0$$




There is no general formula to solve a polynomial equation with degree 5 or above on $\mathbb{R}$. However, we may approximate the solution by an iterative method.

Let

$$g(x) = x^5 - x - 2$$

We first estimate the interval for which the root of $g(x) = 0$ lies in:
Observe that

$$g(1) = -2, \quad g(2) = 28$$

By intermediate value theorem, there is a root $z$ lying in the interval $(1, 2)$.

In order to approximate the root $z$ with certain accuracy, we may define an iterative scheme. Before introducing an iterative scheme, we may observe the following:

We can rewrite $g(x) = x^5 - x - 2 = 0$ as

$$x = (x+2)^{\frac{1}{5}} \quad \text{or} \quad x = x^5 - 2$$

**Question**

(a) Let $f(x) = (x+2)^{\frac{1}{5}}$. Find $f'(x)$.

(b) Let $f(x) = x^5 - 2$. Find $f'(x)$.

After calculation, you may notice that for $x \in (1, 2)$,

$$\text{(a) } |f'(x)| \leq \frac{1}{5} \qquad\qquad \text{(b) } |f'(x)| \geq 5$$

By exercise 3, if we choose the definition of $f$ in (a), the sequence

$$x_{n+1} = f(x_n)$$

will converges to $z$ that satisfies $x = f(x)$. That is,

$$z = f(z) = (z+2)^{\frac{1}{5}}$$

Recall that $z$ is the root of $g$. In other words, the sequence $\{x_n\}$ converges to the root of $g$.

Hence we come up with an iterative scheme:

$$x_1 \in (1,2), \quad x_{n+1} = f(x_n) = (x_n+2)^{\frac{1}{5}}$$

which will converge to an approximate solution to $g(x) = x^5 - x - 2 = 0$.

Here is an example

$$\begin{gathered} \text{Choose } x_1 = 1.5. \text{ Then} \\ x_2 = f(x_1) \approx 1.2847351571 \\ x_3 = f(x_2) \approx 1.2685280409 \\ x_4 = f(x_3) \approx 1.2672737615 \\ \vdots \\ x_{1000} = f(x_{999}) \approx 1.2671683045 \end{gathered}$$

By computation,

$$g(x_{1000}) \approx -4.44089 \times 10^{-16}$$

Therefore, we have a well-approximated solution.

If we take $f$ in (b) as our definition, we have the following observation.

$$x_1 \in (1,2), \quad x_{n+1} = f(x_n) = x_n^5 - 2$$

$$\begin{gathered} \text{Choose } x_1 = 1.5. \text{ Then} \\ x_2 = f(x_1) = 5.59375 \\ x_3 = f(x_2) \approx 5.475 \times 10^3 \\ x_4 = f(x_3) \approx 4.918 \times 10^{18} \\ x_5 = f(x_4) \approx 2.877 \times 10^{93} \\ \vdots \end{gathered}$$

Indeed, $f$ is strictly increasing for $x \in [1, \infty)$ by our computation on its derivative, and the sequence does not converge to our solution. Therefore, if we take this definition, the iterative scheme fails.

Remark: $f$ must be a well-defined function on $\mathbb{R}$ so that the iterative scheme works.

**Solution**

**Exercise 1**:

(a) Observe that when $x = 0$, the inequality holds.

Let $f(u) = \log(1-u)$. Let $x \in (0,1)$.

Observe that $f$ is continuous on $[0,x]$ and differentiable on $(0,x)$, with

$$f'(u) = -\frac{1}{1-u} \leq -1 \text{ for any } u \in (0,1)$$

By (Lagrange) Mean Value Theorem, there exists $\xi \in (0,x)$ such that

$$\frac{\log(1-x)}{x} = \frac{f(x)-f(0)}{x-0} = f'(\xi) \leq -1$$

Therefore,

$$\log(1-x) \leq -x$$

(b) Observe that when $x = 0$, the inequality holds.

Let $f(u) = \log(1-u) + u^2$. Let $x \in \left(0, \frac{1}{2}\right]$.

Observe that $f$ is continuous on $[0,x]$ and differentiable on $(0,x)$, with

$$f'(u) = -\frac{1}{1-u} + 2u \text{ for any } u \in \left(0, \frac{1}{2}\right)$$

Observe that $2(1-u) \geq 1$. Then $2u \geq \frac{u}{1-u} = -1 + \frac{1}{1-u}$. Hence $f'(u) \geq -1$.

By (Lagrange) Mean Value Theorem, there exists $\xi \in (0,x)$ such that

$$\frac{\log(1-x) + x^2}{x} = \frac{f(x)-f(0)}{x-0} = f'(\xi) \geq -1$$

Therefore,

$$\log(1-x) \geq -x - x^2$$

(c) Let $c \in [0,1]$, $x \in [0,1]$.

Observe that when $(c,x) = (0,0), (0,1), (1,0)$ or $(1,1)$, the inequality holds.

We exclude the above cases and further let $c \in (0,1), x \in (0,1)$.

Originally, we want to show

$$(1-c)^x \leq 1 - cx$$

Interchanging $c$ and $x$, we have

$$(1-x)^c \leq 1 - cx$$

Let $f(u) = (1-u)^c$.

Observe that $f$ is continuous on $[0,x]$ and differentiable on $(0,x)$, with

$$f'(u) = -c(1-u)^{c-1} \text{ for any } u \in (0,1)$$

Observe that $f'(u) = -c\frac{1}{(1-u)^{1-c}} \leq -c$ (Verify it).

By (Lagrange) Mean Value Theorem, there exists $\xi \in (0,x)$ such that

$$\frac{(1-x)^c - 1}{x} = \frac{f(x)-f(0)}{x-0} = f'(\xi) \leq -c$$

Therefore,

$$(1-x)^c \le 1-cx$$

Interchanging $x$ and $c$ again, we get

$$(1-c)^x \le 1-cx$$

**Exercise 2**:

(a) Note that $f$ is continuous on $[x, x+1]$ and differentiable on $(x, x+1)$.
By (Lagrange) Mean Value Theorem, there exists $\xi \in (x, x+1)$ such that

$$f(x+1) - f(x) = \frac{f(x+1) - f(x)}{(x+1) - x} = f'(\xi)$$

Since $f'$ is strictly increasing,

$$f'(x) < f'(\xi) < f'(x+1)$$

Therefore,

$$f'(x) < f(x+1) - f(x) < f'(x+1)$$

(b) For $n \in \mathbb{N} \setminus \{1\}$,

$$\begin{aligned} f(n) - f(1) &= \Big(f(n) - f(n-1)\Big) + \Big(f(n-1) - f(n-2)\Big) + ... + \Big(f(3) - f(2)\Big) + \Big(f(2) - f(1)\Big) \\ &= \sum_{m=1}^{n-1} \Big(f(m+1) - f(m)\Big) \end{aligned}$$

By (a), for $m = 1, 2, ..., n-1$,

$$f'(m) < f(m+1) - f(m) < f'(m+1)$$

Summing all the terms,

$$\sum_{m=1}^{n-1} f'(m) < \sum_{m=1}^{n-1} \Big(f(m+1) - f(m)\Big) < \sum_{m=1}^{n-1} f'(m+1)$$

Therefore,

$$f'(1) + f'(2) + ... + f'(n-1) < f(n) - f(1) < f'(2) + f'(3) + ... + f'(n)$$

**Exercise 3(1)**:

Let $h(x) = f(x) - x$. Suppose not, $f(x) \neq x$ for any $x \in \mathbb{R}$.

There are three cases.

1 There exists $x, y \in \mathbb{R}$ such that $f(x) - x < 0$ and $f(y) - y > 0$.

2 $f(x) - x > 0$ for all $x \in \mathbb{R}$.

3 $f(x) - x < 0$ for all $x \in \mathbb{R}$.

**Case 1**
Observe that $h(x) < 0$ and $h(y) > 0$. $f$ is differentiable, and hence continuous.
Therefore $h$ is continuous. By intermediate value theorem, there exists $z$ between $x$ and $y$ such that

$$h(z) = 0$$

Then $f(z) = z$, which leads to contradiction.

**Case 2**
We have $f(0) > 0$.
Since $f$ is differentiable on $\mathbb{R}$, $h$ is differentiable on $\mathbb{R}$.
Let $x > 0$. Note that $h$ is continuous on $[0, x]$ and differentiable on $(0, x)$.
By mean value theorem, there exists $\xi \in (0, x)$ such that

$$\frac{f(x) - f(0)}{x - 0} = f'(\xi) \leq M < 1$$

Then

$$f(x) < x + f(0)$$

By our assumption,

$$x < f(x) < x + f(0)$$
$$1 < \frac{f(x)}{x} < 1 + \frac{f(0)}{x}$$

Note that $\lim_{x \to \infty} 1 + \frac{f(0)}{x} = \lim_{x \to \infty} 1 = 1$.

By squeeze theorem,

$$\lim_{x \to \infty} \frac{f(x)}{x} = 1$$

Therefore,

$$\lim_{x \to \infty} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to \infty} \left( \frac{f(x)}{x} - \frac{f(0)}{x} \right) = 1$$

Earlier we showed that

$$\frac{f(x) - f(0)}{x - 0} \leq M < 1$$

Letting $x \to \infty$,

$$1 = \lim_{x \to \infty} \frac{f(x) - f(0)}{x - 0} \leq M < 1$$

which leads to contradiction.

**Case 3**
By using similar argument in case 2, we can show that it is not possible.

All the cases are not possible. Therefore, $f(x) = x$ for some $x \in \mathbb{R}$.

**Exercise 3(2)**:

Suppose not, there were more than one $z$ that satisfies $f(x) = x$.

Let $z_1, z_2$, where $z_1 \neq z_2$, be solutions to $f(x) = x$. That is,

$$f(z_1) = z_1, \quad z_2 = f(z_2)$$

Since $f$ is continuous inclusively between $z_1$ and $z_2$, and is differentiable exclusively between $z_1$ and $z_2$, by mean value theorem, there exists $\xi$ between $z_1$ and $z_2$ such that

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = f'(\xi) \leq M < 1$$

However,

$$\frac{f(z_1) - f(z_2)}{z_1 - z_2} = \frac{z_1 - z_2}{z_1 - z_2} = 1$$

which leads to contradiction.
Therefore, there is at most one solution.

**Exercise 3(3)**:

**Case 1**
Suppose $x_k = z$ for some $k \in \mathbb{N}$.
Observe that

$$x_{k+1} = f(x_k) = f(z) = z$$

Then we can show, inductively, that $x_n = z$ for all $n \geq k$.

Therefore, $\lim_{n\to\infty} x_n = z$.

**Case 2**
Suppose $x_k \neq z$ for all $k \in \mathbb{N}$.
Let $m = 1, 2, 3, ..., n-1$.
Note that $f$ is continuous inclusively between $z$ and $x_m$.
Also, $f$ is differentiable exclusively between $z$ and $x_m$ .
By (Lagrange) mean value theorem, there exists $\xi_m$ exclusively between $z$ and $x_m$ such that

$$\frac{f(x_m) - f(z)}{x_m - z} = f'(\xi_m)$$

Hence, by our assumption,

$$\left|\frac{f(x_m) - f(z)}{x_m - z}\right| = \left|f'(\xi_m)\right| \leq M$$

Then

$$\begin{aligned}
\left|x_n - z\right| &= \left|f(x_{n-1}) - f(z)\right| \\
&= \left|\frac{f(x_{n-1}) - f(z)}{x_{n-1} - z}\,(x_{n-1} - z)\right| \\
&= \left|\frac{f(x_{n-1}) - f(z)}{x_{n-1} - z}\right| \left|x_{n-1} - z\right| \\
&\leq M \left|x_{n-1} - z\right| \\
&\leq M^2 \left|x_{n-2} - z\right| \\
&\leq M^{n-1} \left|x_1 - z\right|
\end{aligned}$$

Since $M < 1$, $\lim_{n\to\infty} M^{n-1} \left|x_1 - z\right| = 0$.

By squeeze theorem, $\lim_{n\to\infty} x_n - z = 0$.

Therefore, $\lim_{n\to\infty} x_n = z$.