THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics (Spring 2018)
Tutorial 12 (Supplementary)
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Exercise 1:

Evaluate the following integrals by using partial fraction.

(a) $\displaystyle\int \frac{-3x+4}{(x+1)(x-6)}dx$

(b) $\displaystyle\int \frac{1}{x(x^2-2x+2)}dx$

Exercise 2:

Let x be a positive real number. Show that

$$\int_0^{\frac{1}{x^3}} \sin\sqrt{xt}dt = \frac{1}{x}\int_0^{\frac{1}{x^2}} \sin\sqrt{t}dt$$

and hence evaluate

$$\lim_{x\to\infty} x^4 \int_0^{\frac{1}{x^3}} \sin\sqrt{xt}dt$$

Exercise 3:

Let $0 \leq x \leq 1$.

Show that

$$\int_0^x \frac{t^{4n}}{1+t^4}dt \leq \frac{x^{4n+1}}{4n+1}$$

and hence

$$\lim_{x\to\infty} \int_0^x \frac{t^{4n}}{1+t^4}dt = 0$$

Solution

Exercise 1:

(a) Let

$$\frac{-3x+4}{(x+1)(x-6)} = \frac{A}{x+1} + \frac{B}{x-6}$$

Then we have

$$-3x+4 = A(x-6) + B(x+1)$$
$$A+B = -3, -6A+B = 4$$

By solving the two equations, $A = -1, B = -2$. Hence,

$$\int \frac{-3x+4}{(x+1)(x-6)}dx = \int \left(-\frac{1}{x+1} - \frac{2}{x-6}\right)dx = -\ln|x+1| - 2\ln|x-6| + C$$

(b) Let

$$\frac{1}{x(x^2-2x+2)} = \frac{A}{x} + \frac{Bx+D}{x^2-2x+2}$$
$$\frac{1}{x(x^2-2x+2)} = \frac{A(x^2-2x+2)+(Bx+D)x}{x(x^2-2x+2)}$$

By solving,

$$A = \frac{1}{2}, B = -\frac{1}{2}, D = 1$$

Therefore,

$$\begin{aligned}
\int \frac{1}{x(x^2-2x+2)}dx &= \int \frac{1}{2x} + \frac{-\frac{1}{2}x+1}{x^2-2x+2}dx \\
&= \frac{1}{2}\ln|x| + \int \frac{1}{2} \cdot \frac{-x+2}{(x-1)^2+1}dx \\
&= \frac{1}{2}\ln|x| + \frac{1}{2}\int \frac{-(x-1)+1}{(x-1)^2+1}dx \\
&= \frac{1}{2}\ln|x| - \frac{1}{2}\int \frac{x-1}{(x-1)^2+1}dx + \frac{1}{2}\int \frac{1}{(x-1)^2+1}dx \\
&= \frac{1}{2}\ln|x| - \frac{1}{4}\ln|(x-1)^2+1| + \frac{1}{2}\tan^{-1}(x-1) + C
\end{aligned}$$

Exercise 2:

Let $u = xt$. Then

$$\int_0^{\frac{1}{x^3}} \sin\sqrt{xt}dt = \int_0^{\frac{1}{x^2}} \sin\sqrt{u}\frac{1}{x}du = \frac{1}{x}\int_0^{\frac{1}{x^2}} \sin\sqrt{u}du = \frac{1}{x}\int_0^{\frac{1}{x^2}} \sin\sqrt{t}dt$$

Then

$$\begin{aligned}\lim_{x\to\infty} x^4 \int_0^{\frac{1}{x^3}} \sin\sqrt{xt}dt &= \lim_{x\to\infty} x^3 \int_0^{\frac{1}{x^2}} \sin\sqrt{t}dt \\ &= \lim_{x\to\infty} \frac{\displaystyle\int_0^{\frac{1}{x^2}} \sin\sqrt{t}dt}{\frac{1}{x^3}}\end{aligned}$$

As $\frac{1}{x^3} \to 0$ and $\int_0^{\frac{1}{x^2}} \sin\sqrt{t}dt \to 0$, by L'Hospital's Rule,

$$\begin{aligned}\lim_{x\to\infty} x^4 \int_0^{\frac{1}{x^3}} \sin\sqrt{xt}dt &= \lim_{x\to\infty} \frac{-\frac{2}{x^3}\sin\sqrt{\frac{1}{x^2}}}{-\frac{3}{x^4}} \\ &= \lim_{x\to\infty} \frac{2\cdot\sin\frac{1}{x}}{3\cdot\frac{1}{x}} \\ &= \frac{2}{3}\end{aligned}$$

Exercise 3:

For $0 \le t \le x \le 1$,

$$\frac{t^{4n}}{1+t^4} \le t^{4n}$$

$$\int_0^x \frac{t^{4n}}{1+t^4}dt \le \int_0^x t^{4n}dt = \frac{x^{4n+1}}{4n+1}$$

Note that

$$\frac{t^{4n}}{1+t^4} \ge 0$$

and

$$0 \le \int_0^x \frac{t^{4n}}{1+t^4} \le \frac{x^{4n+1}}{4n+1}$$

Observe that

$$\lim_{n\to\infty} \frac{x^{4n+1}}{4n+1} = 0$$

By Squeeze Theorem,

$$\lim_{n\to\infty} \int_0^x \frac{t^{4n}}{1+t^4} = 0$$