THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 University Mathematics 2017-2018
Suggested Solution to Assignment 7

1. (a) Let $x = \sec u$, then $dx = \tan u \sec u \, du$.
$\int \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int \frac{\tan u \sec u}{\sec^2 u \tan u}\,du = \int \cos u\,du = \sin u + C = \frac{\sqrt{x^2-1}}{x} + C$

(b) $\int \frac{x^3-3x-2}{x^2+x}\,dx = \int x - 1 - \frac{2}{x}\,dx = \frac{x^2}{2} - x - 2\ln|x| + C$

(c) $\int \frac{2x+1}{x^3-1}\,dx = \int \frac{1}{x-1} - \frac{x}{x^2+x+1}\,dx = \int \frac{1}{x-1} - \frac{1}{2}\cdot\frac{2x+1}{x^2+x+1} + \frac{1}{2}\cdot\frac{1}{x^2+x+1}\,dx =$
$\ln|x-1| - \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C$

(d) $\int \frac{6x+11}{(x+1)^2}\,dx = \int \frac{6}{x+1} + \frac{5}{(x+1)^2}\,dx = 6\ln|x+1| - \frac{5}{x+1} + C$

2. (a) $\frac{u^4(1-u)^4}{1+u^2} = (u^6 - 4u^5 + 5u^4 - 4u^2 + 4) - \frac{4}{u^2+1}$
Therefore,

$$\begin{aligned}\int_0^1 \frac{u^4(1-u)^4}{1+u^2}\,du &= \int_0^1 (u^6 - 4u^5 + 5u^4 - 4u^2 + 4) - \frac{4}{u^2+1}\,du \\ &= [\frac{u^7}{7} - \frac{2u^6}{3} + u^5 - \frac{4u^3}{3} + 4u)]_0^1 - [4\tan^{-1} u)]_0^1 \\ &= \frac{22}{7} - \pi\end{aligned}$$

(b)

$$\begin{aligned}\int_0^1 u^4(1-u)^4\,du &= \int_0^1 (u^8 - 4u^7 + 6u^6 - 4u^5 + u^4)\,du \\ &= [\frac{u^9}{9} - \frac{4u^8}{8} + \frac{6u^7}{7} - \frac{4u^6}{6} + \frac{u^5}{5}]_0^1 \\ &= \frac{1}{630}\end{aligned}$$

When $u \in (0,1), 1 < u^2 + 1 < 2$,
so $\frac{u^4(1-u)^4}{2} < \frac{u^4(1-u)^4}{1+u^2} < u^4(1-u)^4$
Hence $\int_0^1 \frac{u^4(1-u)^4}{2}\,du < \int_0^1 \frac{u^4(1-u)^4}{1+u^2} < \int_0^1 u^4(1-u)^4\,du$
By above, $\frac{1}{1260} < \frac{22}{7} - \pi < \frac{1}{630}$
Therefore, $\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}$

3. (a) With respect to the partition $1 < 2 < ... < n$, the minimum and maximum Riemann sum of $f(x)$ are $\sum_{i=1}^{n-1} f(i)$ and $\sum_{i=2}^{n} f(i)$ respectively, since f is increasing. Then by the definition of the Riemann integral, we get the required inequality.

(b) Note that $f(x) = \ln x$ is an increasing function, so by part(a), we get that

$$\ln[(n-1)!] \leq \int_1^n \ln x\,dx \leq \ln(n!)$$

Taking exponential of this inequality yields the second required inequality. The fact that $n^n e^{-n+1} \leq n!$ implies that $(n!)^{1/n}/n \geq e^{-1+1/n}$; and the fact that $n! \leq (n+1)^{n+1}e^{-n}$ implies that $(n!)^{1/n}/n \leq (n+1)^{1+1/n}n^{-1}e^{-1}$,and further calculation shows $\lim_{n\to\infty}(n+1)^{1+1/n}n^{-1}e^{-1} = 1/e$. Hence

$$\lim_{n\to\infty}\frac{(n!)^{1/n}}{n} = 1/e$$

4. (a) Using integration by parts,

$$\begin{aligned}\int_0^x (x-t)^p f'(t)\,dt &= \int_0^x (x-t)^p\, d(f(t)) \\ &= [(x-t)^p f(t)]_0^x - \int_0^x f(t)d(x-t)^p \\ &= -x^p f(0) + p\int_0^x f(t)(x-t)^{p-1}\,dt\end{aligned}$$

(b) Let $f(t) = e^t$, then $f'(t) = e^t$ and $f(0) = 1$. Therefore, by (a),

$$\int_0^x (x-t)^p e^t\,dt = -x^p + p\int_0^x (x-t)^{p-1}e^t\,dt.$$

Repeatly using the above,

$$\begin{aligned}\frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}e^t\,dt &= -\frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-2)!}\int_0^x (x-t)^{n-2}e^t\,dt \\ &= \frac{x^{n-1}}{(n-1)!} - \frac{x^{n-2}}{(n-2)!} + \frac{1}{(n-3)!}\int_0^x (x-t)^{n-3}e^t\,dt \\ &\vdots \\ &= \frac{x^{n-1}}{(n-1)!} - \frac{x^{n-2}}{(n-2)!} - \cdots - \frac{x}{1!} + \frac{1}{0!}\int_0^x e^t\,dt \\ &= \frac{x^{n-1}}{(n-1)!} - \frac{x^{n-2}}{(n-2)!} - \cdots - \frac{x}{1!} - 1 + e^x \\ e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} + \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}e^t\,dt.\end{aligned}$$

Replacing n by $2n+1$. If we put $x = 1$, we have

$$e - \left(1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{(2n)!}\right) = \frac{1}{(2n)!}\int_0^1 (1-t)^{2n}e^t\,dt.$$

If we put $x = -1$, we have

$$\frac{1}{e} - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + \frac{1}{(2n)!}\right) = \frac{1}{(2n)!}\int_0^{-1}(-1-t)^{2n}e^t\,dt = \frac{1}{(2n)!}\int_0^1 (1-t)^{2n}e^{-t}\,dt.$$

(The last equality can be obtained by using substitution $u = -t$.)

Therefore,

$$\left(e + \frac{1}{e}\right) - 2\left(1 + \frac{1}{2!} + \frac{1}{4!} + \cdots + \frac{1}{(2n)!}\right) = \frac{1}{(2n)!}\int_0^1 (1-t)^{2n}(e^t + e^{-t})\,dt.$$

If $g(t) = e^t + e^{-t}$, $g'(t) = e^t - e^{-t}$. Thus, $g'(t) > 0$ for $0 < t < 1$. Also, $g(t)$ is continuous at 0 and 1, so $g(t)$ is strictly increasing on $[0,1]$ and $0 = g(0) \leq g(t) \leq g(1) = 3$ for $t \in [0,1]$.

On the other hand, $0 \leq (1-t)^{2n} \leq 1$ for $t \in [0,1]$.

As a result, $0 \le (1-t)^{2n}(e^t - e^{-t}) \le 3$ for $t \in [0,1]$.

Then, for $0 \le t \le 1$,

$$\begin{aligned}
\left|\left(e+\frac{1}{e}\right) - 2\left(1+\frac{1}{2!}+\frac{1}{4!}+\cdots+\frac{1}{(2n)!}\right)\right| &= \left|\frac{1}{(2n)!}\int_0^1 (1-t)^{2n}(e^t+e^{-t})\,dt\right| \\
&\le \frac{1}{(2n)!}\int_0^1 \left|(1-t)^{2n}(e^t+e^{-t})\right|\,dt \\
&= \frac{1}{(2n)!}\int_0^1 (1-t)^{2n}(e^t+e^{-t})\,dt \\
&\le \frac{1}{(2n)!}\int_0^1 3\,dt \\
&= \frac{3}{(2n)!}
\end{aligned}$$

5. (a) $\displaystyle\int_0^1 (1-x)f'(x)\,dx = \int_0^1 1-x\,d(f(x)) = [(1-x)f(x)]_0^1 + \int_0^1 f(x)\,dx = f(0)+I$ and
$\displaystyle\int_0^1 xf'(x)\,dx = \int_0^1 x\,d(f(x)) = [xf(x)]_0^1 - \int_0^1 f(x)\,dx = f(1)-I.$
The result follows by rearranging the terms.

By the above, we have

$$\begin{aligned}
2I &= f(0)+f(1)+\int_0^1 (1-x)f'(x)\,dx - \int_0^1 xf'(x)\,dx \\
I &= \frac{f(0)+f(1)}{2} + \frac{1}{2}\int_0^1 (1-2x)f'(x)\,dx \\
&= \frac{f(0)+f(1)}{2} + \frac{1}{2}\int_0^1 f'(x)\,d(x-x^2) \\
&= \frac{f(0)+f(1)}{2} + \frac{1}{2}\left([(x-x^2)f'(x)]_0^1 - \int_0^1 (x-x^2)\,d(f'(x))\right) \\
&= \frac{f(0)+f(1)}{2} - \frac{1}{2}\int_0^1 x(1-x)f''(x)\,dx
\end{aligned}$$

(b) Note that

$$\left|I - \frac{f(0)+f(1)}{2}\right| = \left|\frac{1}{2}\int_0^1 (1-2x)f'(x)\,dx\right| \le \frac{1}{2}\int_0^1 |(1-2x)f'(x)|\,dx \le \frac{K}{2}\int_0^1 |1-2x|\,dx = \frac{K}{4}.$$

Also,

$$\left|I - \frac{f(0)+f(1)}{2}\right| = \left|\frac{1}{2}\int_0^1 x(1-x)f''(x)\,dx\right| \le \frac{1}{2}\int_0^1 |x(1-x)f''(x)|\,dx \le \frac{M}{2}\int_0^1 |x(1-x)|\,dx = \frac{K}{12}.$$

Therefore, $\left|I - \dfrac{f(0)+f(1)}{2}\right| \le \min\{\dfrac{M}{4}, \dfrac{K}{12}\}$.

(Remark: $\dfrac{f(0)+f(1)}{2}$ is the area of the trapezium with vertices $(0,0)$, $(0,f(0))$, $(1,f(1))$ and $(1,0)$. The above result shows that if we use the area of the trapezium to approximate I, then the absolute error is less than $\min\{\dfrac{M}{4}, \dfrac{K}{12}\}$.)

6. (a) i.

$$I_0 = \int_0^{\frac{\pi}{2}} \sin x\, dx = [-\cos x]_0^{\frac{\pi}{2}} = 1$$

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx \\ &= -\int_0^{\frac{\pi}{2}} \sin^{2n} x\, d\cos x \\ &= [-\sin^{2n} x \cos x]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x\, d\sin^{2n} x \\ &= 0 + \int_0^{\frac{\pi}{2}} 2n \cos^2 x \sin^{2n-1} x\, dx \\ &= \int_0^{\frac{\pi}{2}} 2n(1-\sin^2 x)\sin^{2n-1} x\, dx \\ &= 2n\int_0^{\frac{\pi}{2}} \sin^{2n-1} x\, dx - 2n\int_0^{\frac{\pi}{2}} \sin^{2n+1} x\, dx \\ &= 2nI_{n-1} - 2nI_n \end{aligned}$$

Therefore,

$$I_n = \frac{2n}{2n+1} I_{n-1}$$

ii. When $n = 0$, LHS$= I_0 = 1$, RHS$= 1$, so the statement holds for $n = 0$

Assume the statement holds for $n = k$, Then

$$\begin{aligned} I_{k+1} &= \frac{2k+2}{2k+3} I_k = \frac{2k+2}{2k+3} \cdot \frac{(k!)^2 2^{2k}}{(2k+1)!} = \frac{(2\cdot(k+1))^2}{2k+3} \cdot \frac{(k!)^2 2^{2k}}{(2k+2)(2k+1)!} \\ &= \frac{((k+1)!)^2 2^{2(k+1)}}{(2(k+1)+1)!} \end{aligned}$$

Hence the statement is true for $n = k+1$. By mathematical induction, the statement holds for all n.

(b) i.

$$\begin{aligned} S_m &= \sum_{n=0}^{m} \frac{I_n}{2^{n-1}} \\ &= \int_0^{\frac{\pi}{2}} 2\sin x(1 + \frac{\sin^2 x}{2} + \frac{\sin^4 x}{2^2} + \cdots + \frac{\sin^{2m} x}{2m})\, dx \\ &= \int_0^{\frac{\pi}{2}} 2\sin x \frac{1-(\frac{1}{2}\sin^2 x)^{m+1}}{1-\frac{1}{2}\sin^2 x}\, dx \end{aligned}$$

ii.

$$S_m = \int_0^{\pi/2} \frac{2\sin x}{1-\frac{1}{2}\sin^2 x}\, dx - \int_0^{\frac{\pi}{2}} 2\sin x \frac{(\frac{1}{2}\sin^2 x)^{m+1}}{1-\frac{1}{2}\sin^2 x}\, dx$$

$$\int_0^{\frac{\pi}{2}} 2\sin x \frac{(\frac{1}{2}\sin^2 x)^{m+1}}{1-\frac{1}{2}\sin^2 x}\, dx = \frac{1}{2^m}\int_0^{\frac{\pi}{2}} 2\sin x \frac{\left(\sin^2 x\right)^{m+1}}{2-\sin^2 x} dx$$

$$\because 0 \le (\sin^2 x)^{m+1} \le 1, 2 - \sin^2 x \ge 1$$

$$\therefore 0 \le \frac{\left(\sin^2 x\right)^{m+1}}{2-\sin^2 x} \le 1$$

$$\therefore 0 \leq \int_0^{\frac{\pi}{2}} 2\sin x \frac{(\frac{1}{2}\sin^2 x)^{m+1}}{1-\frac{1}{2}\sin^2 x}\,dx \leq \frac{1}{2^m}\int_0^{\frac{\pi}{2}} 2\sin x\,dx \leq \frac{\pi}{2^m}$$

Therefore,

$$\int_0^{\pi/2} \frac{2\sin x}{1-\frac{1}{2}\sin^2 x}\,dx - \frac{\pi}{2^m} \leq S_m \leq \int_0^{\pi/2} \frac{2\sin x}{1-\frac{1}{2}\sin^2 x}\,dx.$$

Hence,

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^{n+1}}{(2n+1)!} \leq \lim_{n\to\infty} \int_0^{\frac{\pi}{2}} \frac{2\sin x}{1-\frac{1}{2}\sin^2 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{4\sin x}{1+\cos^2 x}\,dx = \pi$$

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^{n+1}}{(2n+1)!} \geq \lim_{n\to\infty} (\int_0^{\frac{\pi}{2}} \frac{2\sin x}{1-\frac{1}{2}\sin^2 x}\,dx - \frac{\pi}{2^n}) = \int_0^{\frac{\pi}{2}} \frac{4\sin x}{1+\cos^2 x}\,dx - \lim_{n\to\infty} \frac{\pi}{2^n} = \pi$$

Therefore,

$$\sum_{n=0}^{\infty} \frac{(n!)^2 2^{n+1}}{(2n+1)!} = \pi$$

7. (a) i. $I_0 = \pi/2$, $I_1 = 1$

   ii. Similar as in 6(a)(i), applying by-part twice to I_{n+2} gives that $I_{n+2} = \frac{n+1}{n+2} I_n$. Therefore

$$I_{2m} = \frac{\pi \cdot 1 \cdot 3 \cdot \ldots \cdot (2m-1)}{2 \cdot 2 \cdot \ldots \cdot 2m}$$

   and

$$I_{2m-1} = \frac{2 \cdot 2 \cdot \ldots \cdot (2m-2)}{1 \cdot 3 \cdot \ldots \cdot (2m-1)}$$

   (b) For $x \in [0, \frac{\pi}{2}]$, $1 \geq \cos x \geq 0$ and hence $\cos^{n+1} x \leq cos^n x$ for any positive integer n. This implies the required inequalities.

   (c) i. After substituting the known formulas for I_n into the inequalities in (b), it is straightforward to get the required inequalities.

   ii. Compute that $A_{n+1}/A_n = \frac{(2n+2)^2}{(2n+1)(2n+3)} > 1$, this means A_n is monotonically increasing.

   iii. The existence of the limit is guaranteed as A_n is bounded and monotonically increasing. By (c)(i), we get that the limit should be $\sqrt{\frac{\pi}{2}}$