THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H/I/J University Mathematics 2017-2018
Assignment 3
Due Date: 23 Feb 2018 (Friday)

1. Evaluate each of the following limits.

   (a) $\lim_{x \to 1} \frac{1-x}{2-\sqrt{x^2+3}}$

   (b) $\lim_{x \to \pi} \frac{\sin x}{x-\pi}$

   (c) $\lim_{x \to 0} \frac{\sin 3x}{\tan 6x}$

   (d) $\lim_{x \to +\infty} \sqrt{4x^2+x+1} - 2x$

   (e) $\lim_{x \to +\infty} \left(\frac{x+3}{x-2}\right)^x$

2. If $f(x) = \frac{|x-2|}{x^2-4}$, evalutate

   (a) $\lim_{x \to 2^-} f(x)$

   (b) $\lim_{x \to 2^+} f(x)$

   (c) $\lim_{x \to -2} f(x)$

3. Evaluate the following limits by using sandwich theorem.

   (a) $\lim_{x \to 4^+} \sqrt{x-4} \cos\left(\frac{1}{\sqrt{x-4}}\right)$

   (b) $\lim_{x \to +\infty} \frac{e^{\cos x}}{x}$

   (c) $\lim_{x \to +\infty} \frac{\cos(\tan x) - \tan(\cos x)}{2x+1}$

4. Suppose that $f(0) = 3$, $g(0) = 4$, $\lim_{x \to 0} \frac{f(x)}{x} = 2$ and $\lim_{x \to 0} \frac{g(x)}{\sin x} = 1$. Find

   (a) $\frac{f(0)}{g(0)}$

   (b) $\lim_{x \to 0} \frac{f(x)}{g(x)}$

   (c) $\lim_{x \to 0} f(x)$

   (d) $\lim_{x \to 0} g(x)$.

5. Let a be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} e^{\frac{1}{2x}} & \text{if} \quad x < 0; \\ 1 & \text{if} \quad x = 0; \\ e^x - a & \text{if} \quad x > 0 \end{cases}$$

(a) If $\lim_{x\to 0} f(x)$ exists, find the value of a.

(b) Is $f(x)$ continuous at $x = 0$?

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} (x-1)\sin(\dfrac{1}{x^2-1}) & \text{if} \quad x \neq 1; \\ \\ 0 & \text{if} \quad x = 1. \end{cases}$$

Show that $f(x)$ is continuous at $x = 1$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that

- f is a positive continuous function;
- $f(\sqrt{x^2 + y^2}) = f(x)f(y)$ for all real numbers x and y.

(a) Show that $f(x) = f(|x|)$ for all real numbers x.

(b) Show that $f(\sqrt{n}x) = [f(x)]^n$ for all real numbers x and positive integers n.

(c) Show that $f(r) = [f(1)]^{r^2}$ for all rational numbers r.

(d) It is known that for all real numbers x, there exists a sequence $\{a_n\}$ of rational numbers such that $\lim_{n\to\infty} a_n = x$.
Show that $f(x) = [f(1)]^{x^2}$ for all real numbers x.