THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010H/I/J University Mathematics 2017-2018
Assignment 1
Due Date: 26 Jan, 2018

1. Consider the function $f(x)$ defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if} \quad x \leq 9, \\ \dfrac{1}{x-9} & \text{if} \quad x > 9. \end{cases}$$

Find the value of $f(4)$, $f(9)$ and $f(16)$.

2. Fill in the blanks:

(a) Consider the function $f(x) = |x|$. The function can be described explicitly by

$$f(x) = \begin{cases} \underline{\hspace{6cm}} & \text{if} \quad x \geq 0, \\ \underline{\hspace{6cm}} & \text{if} \quad x < 0. \end{cases}$$

Hence, sketch the graph of $f(x) = |x|$.

(b) Consider the function $f(x) = |x^2 - 5x + 6|$. The function can be described explicitly by

$$f(x) = \begin{cases} \underline{\hspace{6cm}} & \text{if} \quad x \geq 3, \\ \underline{\hspace{6cm}} & \text{if} \quad 2 < x < 3, \\ \underline{\hspace{6cm}} & \text{if} \quad x \leq 2. \end{cases}$$

Hence, sketch the graph of $f(x) = |x^2 - 5x + 6|$.

3. Sketch the graphs of the following functions.

(a) $f(x) = |2x + 4| + |x - 1|$

(b) $g(x) = \begin{cases} \dfrac{|x|}{x} & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a function.

(a) Show that $\dfrac{f(x) + f(-x)}{2}$ is an even function and $\dfrac{f(x) - f(-x)}{2}$ is an odd function.

(b) Hence, show that $f(x)$ can be expressed as a sum of an even function and an odd function.

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Show that

(a) if f and g are odd functions, then $(f \cdot g)$ is an even function;

(b) if f and g are even functions, then $(f \cdot g)$ is an even function;

(c) if f is an odd function and g is an even function, then $(f \cdot g)$ is an odd function.

6. By using the product to sum formula, express each of the following expressions as a sum of trigonometric functions.

   (a) $\cos 2x \cos 5x$;

   (b) $\sin 3x \sin 7x$;

   (c) $\sin 4x \cos 6x$.

7. Let $t = \tan \frac{x}{2}$, where $-\pi < x < \pi$.

   (a) By considering $\tan x = \tan(2 \cdot \frac{x}{2})$, show that $\tan x = \frac{2t}{1-t^2}$.

   (b) Using (a), express $\sin x$ and $\cos x$ in terms of t.
   Hence, express $\frac{1}{2 + 3\cos x + 4\sin x}$ in terms of t.

   (Remark: We will need this when we cover t-substitution in integration.)

8. Show that

$$2[\cos\theta + \cos(\theta + 2\alpha) + \cos(\theta + 4\alpha) + \cos(\theta + 6\alpha) + \cos(\theta + 8\alpha)]\sin\alpha = \sin(\theta + 9\alpha) - \sin(\theta - \alpha).$$

   Hence, show that

$$\cos\theta + \cos(\theta + \frac{2\pi}{5}) + \cos(\theta + \frac{4\pi}{5}) + \cos(\theta + \frac{6\pi}{5}) + \cos(\theta + \frac{8\pi}{5}) = 0.$$

9. A sequence $\{x_n\}$ is defined by $x_1 = 3$ and $x_{n+1} = 3 + \frac{1}{16}x_n^2$ for $n \geq 1$.

   (a) Prove that $\{x_n\}$ is bounded above by 4, i.e. $x_n \leq 4$ for all positive numbers n.

   (b) Prove that $\{x_n\}$ is an increasing sequence, i.e. $x_{n+1} \geq x_n$ for all positive numbers n.

   (Remark: By the monotone convergence theorem, $\{x_n\}$ is a convergent sequence, i.e. $\lim_{n\to\infty} x_n$ exists.)

10. (a) (Binomial Theorem) Let x and y be real numbers. By using mathematical induction, prove that for all positive numbers n,

$$(x + y)^n = \sum_{r=0}^{n} C_r^n x^r y^{n-r},$$

   where $C_r^n = \frac{n!}{r!(n-r)!}$.

   (b) Hence, expand $(3x - 2)^5$.

11. Show that $\frac{(x+h)^n - x^n}{h} = \sum_{r=1}^{n} C_r^n h^{r-1} x^{n-r}$.

   (Remark: We will need this when we derive the derivative of x^n, where n is a positive integer.)