THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5520
Differential Equations & Linear Algebra
Suggested Solution for Assignment 3
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Exercise 4.2 Question 1(b)

Using the method of reduction of order to solve the equation given that $y_1(t)$ is a solution.

$$t^2y'' + 4ty' + 2y = 0; \quad y_1(t) = t^{-1}.$$

Solution: We set $y(t) = y_1(t)v(t) = t^{-1}v(t)$. Then

$$\begin{cases} y'(t) = t^{-1}v'(t) - t^{-2}v(t), \\ y''(t) = t^{-1}v''(t) - 2t^{-2}v'(t) + 2t^{-3}v(t). \end{cases}$$

Thus the equation becomes

$$\begin{aligned} t^2(t^{-1}v'' - 2t^{-2}v' + 2t^{-3}v) + 4t(t^{-1}v' - t^{-2}v) + 2(t^{-1}v) &= 0 \\ tv'' + 2v' &= 0. \end{aligned}$$

Making use of the substitution $u = v'$, we have

$$u' + 2t^{-1}u = 0.$$

This equation is a first-order linear ODE in u. An integrating factor is

$$\exp\left(\int 2t^{-1}\,dt\right) = \exp(2\ln t) = t^2.$$

Multiplying both sides with t^2, we have

$$\begin{aligned} t^2u' + 2tu &= 0 \\ \frac{d}{dt}(t^2u) &= 0 \\ t^2u &= -c_1 \\ u &= -c_1t^{-2} \\ v' &= -c_1t^{-2} \\ v &= c_1t^{-1} + c_2 \\ y &= t^{-1}(c_1t^{-1} + c_2) \\ y &= c_1t^{-2} + c_2t^{-1}. \end{aligned}$$

Therefore the general solution is $y(t) = c_1t^{-2} + c_2t^{-1}$. □

Exercise 4.3 Question 1(b)

Find the general solution of the following second order linear equations.

$$y'' + 9y = 0.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 + 9 &= 0 \\ r &= \pm 3i. \end{aligned}$$

Thus the general solution is

$$y = c_1 \cos 3t + c_2 \sin 3t.$$

□

Exercise 4.3 Question 1(d)

Find the general solution of the following second order linear equations.

$$y'' - 8y' + 16y = 0.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 - 8r + 16 &= 0 \\ r &= 4. \end{aligned}$$

The characteristic equation has a double root $r = 4$. Thus the general solution is

$$y = c_1 e^{4t} + c_2 t e^{4t}.$$

□

Exercise 4.3 Question 1(e)

Find the general solution of the following second order linear equations.

$$y'' + 4y' + 13y = 0.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 + 4r + 13 &= 0 \\ r &= -2 \pm 3i. \end{aligned}$$

Thus the general solution is

$$y = e^{-2t}(c_1 \cos 3t + c_2 \sin 3t).$$

□

Exercise 4.4 Question 1(e)

Use the method of undetermined coefficients to find the general solution of the following nonhomogeneous second order linear equations.

$$y'' + 2y' + y = 2e^{-t}.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 + 2r + 1 &= 0 \\ r &= -1. \end{aligned}$$

The characteristic equation has a double root $r = -1$. Thus the complementary function is

$$y_c = c_1 e^{-t} + c_2 t e^{-t}.$$

A particular solution is in the form

$$y_p = At^2 e^{-t},$$

where A is a constant to be determined. To find A, we have

$$\begin{cases} y_p' = A(-t^2 + 2t)e^{-t}, \\ y_p'' = A(t^2 - 4t + 2)e^{-t}. \end{cases}$$

By comparing coefficients of

$$\begin{aligned} y_p'' + 2y_p' + y_p &= 2e^{-t} \\ A(t^2 - 4t + 2)e^{-t} + 2A(-t^2 + 2t)e^{-t} + At^2 e^{-t} &= 2e^{-t} \\ 2Ae^{-t} &= 2e^{-t}, \end{aligned}$$

we have $A = 1$ and a particular solution is

$$y_p = t^2 e^{-t}.$$

Therefore the general solution is

$$y = y_c + y_p = c_1 e^{-t} + c_2 t e^{-t} + t^2 e^{-t}.$$

$\square$

Exercise 4.4 Question 1(f)

Use the method of undetermined coefficients to find the general solution of the following nonhomogeneous second order linear equations.

$$y'' - 2y' + y = te^t + 4.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 - 2r + 1 &= 0 \\ r &= 1. \end{aligned}$$

The characteristic equation has a double root $r = 1$. Thus the complementary function is

$$y_c = c_1 e^t + c_2 te^t.$$

A particular solution is in the form

$$y_p = t^2((At + B)e^t) + C = At^3 e^t + Bt^2 e^t + C,$$

where A, B, C are constants to be determined. To find A, B, C, we have

$$\begin{cases} y_p' = A(t^3 + 3t^2)e^t + B(t^2 + 2t)e^t, \\ y_p'' = A(t^3 + 6t^2 + 6t)e^t + B(t^2 + 4t + 2)e^t. \end{cases}$$

By comparing coefficients of

$$\begin{aligned} y_p'' - 2y_p' + y_p &= te^t + 4 \\ (6At + 2B)e^t + C &= te^t + 4, \end{aligned}$$

we have

$$\begin{cases} A = \dfrac{1}{6}, \\ B = 0, \\ C = 4, \end{cases}$$

and a particular solution is

$$y_p = \frac{1}{6} t^3 e^t + 4.$$

Therefore the general solution is

$$y = y_c + y_p = c_1 e^t + c_2 te^t + \frac{1}{6} t^3 e^t + 4.$$

□

Exercise 4.4 Question 2(a)

Write down a suitable form $y_p(t)$ of a particular solution of the following nonhomogeneous second order linear equations.

$$y'' + 3y' = 2t^4 + t^2e^{-3t} + \sin 3t.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 + 3r &= 0 \\ r &= 0, -3. \end{aligned}$$

Thus the complementary function is

$$y_c = c_1 + c_2e^{-3t}.$$

A particular solution is in the form

$$\begin{aligned} y_p = t(A_4t^4 &+ A_3t^3 + A_2t^2 + A_1t + A_0) \\ &+ t(B_2t^2 + B_1t + B_0)e^{-3t} \\ &+ (C_0 \cos 3t + D_0 \sin 3t), \end{aligned}$$

where $A_0, A_1, A_2, A_3, A_4, B_0, B_1, B_2, C_0, D_0$ are constants to be determined. □

Exercise 4.4 Question 2(b)

Write down a suitable form $y_p(t)$ of a particular solution of the following nonhomogeneous second order linear equations.

$$y'' - 5y' + 6y = e^t \cos 2t + 3te^{2t} \sin t.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 - 5r + 6 &= 0 \\ r &= 2, 3. \end{aligned}$$

Thus the complementary function is

$$y_c = c_1e^{2t} + c_2e^{3t}.$$

A particular solution is in the form

$$\begin{aligned} y_p = \ & e^t(A_0 \cos 2t + B_0 \sin 2t) \\ & + e^{2t}((C_1t + C_0) \cos t + (D_1t + D_0) \sin t), \end{aligned}$$

where $A_0, B_0, C_0, C_1, D_0, D_1$ are constants to be determined. □

Exercise 4.4 Question 2(c)

Write down a suitable form $y_p(t)$ of a particular solution of the following nonhomogeneous second order linear equations.

$$y'' + y = t(1 + \sin t).$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 + 1 &= 0 \\ r &= \pm i. \end{aligned}$$

Thus the complementary function is

$$y_c = c_1 \cos t + c_2 \sin t.$$

A particular solution is in the form

$$\begin{aligned} y_p = \ & (A_1 t + A_0) \\ & + t((B_1 t + B_0) \cos t + (C_1 t + C_0) \sin t), \end{aligned}$$

where $A_0, A_1, B_0, B_1, C_0, C_1$ are constants to be determined. □

Exercise 4.5 Question 1(a)

Use the method of variation of parameters to solve the equations.

$$y'' - 5y' + 6y = 2e^t.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 - 5r + 6y &= 0 \\ r &= 2, 3. \end{aligned}$$

Let $y_1 = e^{2t}$ and $y_2 = e^{3t}$. Then complementary function is $y_c = C_1 y_1 + C_2 y_2$. The Wronskian W is given by

$$W = W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = (e^{2t})(3e^{3t}) - (2e^{2t})(e^{3t}) = e^{5t}.$$

Now $g(t) = 2e^t$. Hence

$$\begin{cases} u_1' = -\dfrac{g y_2}{W} = -\dfrac{(2e^t)(e^{3t})}{e^{5t}} = -2e^{-t}, \\ \\ u_2' = \dfrac{g y_1}{W} = \dfrac{(2e^t)(e^{2t})}{e^{5t}} = 2e^{-2t}. \end{cases} \implies \begin{cases} u_1 = 2e^{-t} + c_1, \\ \\ u_2 = -e^{-2t} + c_2. \end{cases}$$

Hence the general solution is

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= (2e^{-t} + c_1)e^{2t} + (-e^{-2t} + c_2)e^{3t} \\ &= c_1 e^{2t} + c_2 e^{3t} + e^t. \end{aligned}$$

□

Exercise 4.5 Question 1(b)

Use the method of variation of parameters to solve the equations.

$$y'' - y' - 2y = 2e^{-t}.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^2 - r - 2 &= 0 \\ r &= -1, 2. \end{aligned}$$

Let $y_1 = e^{-t}$ and $y_2 = e^{2t}$. Then complementary function is $y_c = C_1 y_1 + C_2 y_2$. The Wronskian W is given by

$$W = W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = (e^{-t})(2e^{2t}) - (-e^{-t})(e^{2t}) = 3e^t.$$

Now $g(t) = 2e^{-t}$. Hence

$$\begin{cases} u_1' = -\dfrac{gy_2}{W} = -\dfrac{(2e^{-t})(e^{2t})}{3e^t} = -\dfrac{2}{3}, \\ \\ u_2' = \dfrac{gy_1}{W} = \dfrac{(2e^{-t})(e^{-t})}{3e^t} = \dfrac{2}{3}e^{-3t}. \end{cases} \Longrightarrow \begin{cases} u_1 = -\dfrac{2}{3}t + \left(c_1 + \dfrac{2}{9}\right), \\ \\ u_2 = -\dfrac{2}{9}e^{-3t} + c_2. \end{cases}$$

Hence the general solution is

$$\begin{aligned} y &= u_1 y_1 + u_2 y_2 \\ &= \left(-\frac{2}{3}t + \left(c_1 + \frac{2}{9}\right)\right) e^{-t} + \left(-\frac{2}{9}e^{-3t} + c_2\right) e^{2t} \\ &= c_1 e^{-t} + c_2 e^{2t} - \frac{2}{3} t e^{-t}. \end{aligned}$$

$\square$

Exercise 4.7 Question 1(c)

Write down a suitable form $y_p(t)$ of a particular solution of the following equations.

$$y^{(4)} - 2y'' + y = te^t.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^4 - 2r^2 + 1 &= 0 \\ (r^2)^2 - 2(r^2) + 1 &= 0 \\ r^2 &= 1 \\ r &= \pm 1. \end{aligned}$$

The characteristic equation has two double roots $r = \pm 1$. Thus the complementary function is

$$y_c = c_1 e^{-t} + c_2 te^{-t} + c_3 e^t + c_4 te^t.$$

A particular solution is in the form

$$y_p = \ t^2(A_1 t + A_0)e^t,$$

where A_0, A_1 are constants to be determined. $\square$

Exercise 4.7 Question 1(e)

Write down a suitable form $y_p(t)$ of a particular solution of the following equations.

$$y^{(4)} + 2y'' + y = t \cos t.$$

Solution: Solving the characteristic equation

$$\begin{aligned} r^4 + 2r^2 + 1 &= 0 \\ (r^2)^2 + 2(r^2) + 1 &= 0 \\ r^2 &= -1 \\ r &= \pm i. \end{aligned}$$

The characteristic equation has two double roots $r = \pm i$. Thus the complementary function is

$$y_c = c_1 \cos t + c_2 t \cos t + c_3 \sin t + c_4 t \sin t.$$

A particular solution is in the form

$$y_p = \ t^2((A_1 t + A_0) \cos t + (B_1 t + B_0) \sin t),$$

where A_0, A_1, B_0, B_1 are constants to be determined. $\square$