MMAT 5011 Analysis II
2016-17 Term 2
Assignment 1
Due date: Feb 7, 2017

1. Let $V = M_{3\times 3}(\mathbb{R})$ be the vector space of all 3×3 real matrices. Verify from definition if each of following subsets is a vector subspace of V or not. If it is a vector subspace, write down a basis for it.

   (a) $\{A \in M_{3\times 3}(\mathbb{R}) : \det A = 0\}$;

   (b) $\{(a_{ij}) \in M_{3\times 3}(\mathbb{R}) : a_{ij} \geq 0,\ i,j = 1,2,3\}$;

   (c) $\{(a_{ij}) \in M_{3\times 3}(\mathbb{R}) : a_{ij} = -a_{ji},\ i,j = 1,2,3\}$, the subset of skew- symmetric matrices.

2. Let $n \geq 2$. Verify that the function

$$\|\mathbf{z}\|_2 = \sqrt{\sum_{i=1}^{n} |z_i|^2} \text{ for } \mathbf{z} = (z_1, z_2, \ldots, z_n)$$

   defines a norm on $\mathbb{C}^n$. You can use any inequalities and their finite versions discussed in class. How about

$$\|\mathbf{z}\|_{1/2} = \left(\sum_{i=1}^{n} \sqrt{|z_i|} \right)^2 \text{ for } \mathbf{z} = (z_1, z_2, \ldots, z_n)?$$

3. Let $1 \leq p < q$ and consider the real version of l^p-space

$$l^p = \left\{ \mathbf{x} = (x_1, x_2, \ldots) : \sum_{i=1}^{\infty} |x_i|^p < \infty \right\}.$$

   (a) Show that $l^p \subset l^q$. Two useful fact for real series you may use are

   - If $\sum_{i=1}^{\infty} |a_i| < \infty$, then $\lim_{i\to\infty} a_i = 0$.
   - If $0 \leq a_i \leq b_i$ for all large enough i and $\sum_{i=1}^{\infty} b_i < \infty$, then $\sum_{i=1}^{\infty} a_i < \infty$.

   (b) Show that the inclusion $l^p \subset l^q$ is proper. In other words, find an element $\mathbf{x} = (x_1, x_2, \ldots)$ which lies in l^q but not in l^p.

4. Let $n > 0$ and $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots b_n$ be positive real numbers. Show that

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \ldots + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + \ldots + a_n)^2}{b_1 + b_2 + \ldots + b_n}.$$

   Hence, show that

$$\frac{x^2}{3^3} + \frac{y^2}{4^3} + \frac{z^2}{5^3} \geq \frac{(x+y+z)^2}{6^3}$$

   for $x, y, z > 0$.

5. Let $\mathbf{x}, \mathbf{y} \in l^\infty$. Show that $\mathbf{x} + \mathbf{y} \in l^\infty$ with

$$\|\mathbf{x} + \mathbf{y}\|_\infty \le \|\mathbf{x}\|_\infty + \|\mathbf{y}\|_\infty.$$

6. Show that for any $\mathbf{x}, \mathbf{y} \in l^p$,

$$\big|\|\mathbf{x}\|_p - \|\mathbf{y}\|_p\big| \le \|\mathbf{x} - \mathbf{y}\|_p.$$

7. Since the natural logarithm has second derivatives $(\log x)'' = -\frac{1}{x^2} < 0$ on its domain $(0, \infty)$, it is a concave function. Hence, for any $0 < \alpha < 1$ and $x, y > 0$,

$$\log((1-\alpha)x + \alpha y) \ge (1-\alpha)\log x + \alpha \log y.$$

By using the substitution $x = a^p, y = b^q$ and the fact that the exponential funtion e^x is increasing, prove the Young's inequality, which states that for any $a, b > 0$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q.$$