**MMAT 5011 Analysis II**
**2016-17 Term 2**
**Assignment 6**
**Suggested Solution**

1. Let $q(x) = a + bx \in P_1(\mathbb{R})$, for any $p(x) = c + dx \in P_1(\mathbb{R})$, we have

$$f(p) = p'(0) = d = \langle p, q\rangle = \int_0^1 p(x)q(x)\,dx = \frac{bd}{3} + \frac{ad+bc}{2} + ac$$

Thus

$$(a + \frac{1}{2}b)c + (\frac{1}{3}b + \frac{1}{2}a - 1)d = 0$$

holds for all $c, d \in \mathbb{R}$. We have $a + \frac{1}{2}b = 0, \frac{1}{3}b + \frac{1}{2}a - 1 = 0$, which give $a = -6, b = 12$. Hence $q(x) = -6 + 12x$.

2. For $x \in H_1, y \in H_2$,

$$\langle x, (\alpha T)^* y\rangle = \langle (\alpha T)x, y\rangle = \langle Tx, \overline{\alpha} y\rangle = \langle x, T^*(\overline{\alpha} y)\rangle = \langle x, (\overline{\alpha} T^*)y\rangle.$$

Since $x, y$ are arbitrary, we conclude $(\alpha T)^* = \overline{\alpha} T^*$.

3. i) Let $x \in N(T)$. Then $Tx = 0, \langle x, T^* y\rangle = \langle Tx, y\rangle = 0$ for any $y \in H_2$. Consequently, $x \in [T^*(H_2)]^\perp, N(T) \subset [T^*(H_2)]^\perp$ .

ii) Let $x \in [T^*(H_2)]^\perp$. $\langle Tx, y\rangle = \langle x, T^* y\rangle = 0$ for all $y \in H_2$, thus $Tx = 0, x \in N(T)$. This gives the reverse inclusion of i).

4. (a)

$$T_1^* = \left[\frac{1}{2}(T + T^*)\right]^* = \frac{1}{2}(T + T^*)^* = \frac{1}{2}(T^* + (T^*)^*) = \frac{1}{2}(T^* + T) = T_1.$$

$$T_2^* = \left[\frac{1}{2i}(T - T^*)\right]^* = \overline{\left(\frac{1}{2i}\right)}(T - T^*)^* = \frac{-1}{2i}(T^* - (T^*)^*) = \frac{-1}{2i}(T^* - T) = T_2.$$

Thus $T_1$ and $T_2$ are self-adjoint.

(b) $T_1 + iT_2 = \frac{1}{2}(T + T^*) + i\frac{1}{2i}(T - T^*) = T.$

(c)

$$T = S_1 + iS_2 \quad \text{and} \quad T^* = (S_1 + iS_2)^* = S_1^* - iS_2^* = S_1 - iS_2.$$

Hence $S_1 = \frac{1}{2}(T + T^*) = T_1, S_2 = \frac{1}{2i}(T - T^*) = T_2$.

5. Let $\lambda$ be a eigenvalue of $T$ and $x$ be the corresponding eigenvector. We have

$$\lambda\langle x, x\rangle = \langle \lambda x, x\rangle = \langle Tx, x\rangle = \langle x, T^* x\rangle = \langle x, -Tx\rangle = \langle x, -\lambda x\rangle = -\overline{\lambda}\langle x, x\rangle.$$

Since $x$ is an eigenvector, $\langle x, x\rangle \neq 0$. Thus $\lambda + \overline{\lambda} = 0$ and $\lambda$ is purely imaginary.
To show $T$ has an orthonormal eigenbasis, consider the operator $S = iT$. Then $S^* = -iT^* = iT = S$. Hence $S$ is self-adjoint. By spectral theorem, $S$ has an orthonormal eigenbasis $\{x_1, x_2, \ldots, x_n\}$ and the corresponding eigenvalues $\beta_1, \ldots, \beta_n$ are real. Note

that $T(x_k) = -iS(x_k) = -i\beta_k x_k$, which implies each $x_k$ is also an eigenvector of $T$. Hence, $\{x_1, x_2, \ldots, x_n\}$ is also an eigenbasis for $T$.

Alternatively, one can prove the existence of orthonormal eigenbasis by induction on the dimension of $H$ without using spectral theorem as below.

For $n = 1$, it is trivial. Assume it is true for $n = k$.
For $\dim H = k + 1$, note that $\|T\| = \sup_{\|x\|=1} |\langle Tx, x\rangle|$, and $H$ is finite dimensional, we can find $x \in H, \|x\| = 1$ such that

$$|\langle Tx, x\rangle| = \|T\|.$$

By Cauchy-Schwarz inequality, we also have

$$|\langle Tx, x\rangle| \le \|T(x)\|\|x\| \le \|T\|.$$

The equality holds when $T(x)$ and $x$ are linearly dependent, i.e., there exists $\lambda$ such that $T(x) = \lambda x$. This shows that $x$ is an eigenvector and $\lambda$ the corresponding eigenvalue.
Denote by $x_{k+1} = x, \lambda_{k+1} = \lambda$. Consider $H_0 = \operatorname{span}\{x_{k+1}\}^{\perp}$. Then $\dim H_0 = \dim H - 1 = k$. Also, if $x \in H_0$, then

$$\langle Tx, x_{k+1}\rangle = \langle x, -Tx_{k+1}\rangle = -\langle x, \lambda_{k+1}x_{k+1}\rangle = -\overline{\lambda_{k+1}}\langle x, x_{k+1}\rangle = 0$$

Thus $Tx \in H_0$, $T|_{H_0} : H_0 \to H_0$ is an operator on $H_0$. $T^* = -T$ gives $T|_{H_0}^* = -T|_{H_0}$. By our induction assumption, there exists an orthonormal basis $\{x_1, x_2, \cdots, x_k\}$ of $H_0$ such that $T(x_i) = \lambda_i x_i$ for $i = 1, 2, \cdots, k$. Then $\{x_1, x_2, \cdots, x_{k+1}\}$ is a basis we want.

6. Since $T_n, T$ are bounded, $\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\| \to 0$. Thus $T_n^* \to T^*$.

7. Let $S = I + T^*T$. Suppose $x \in N(S)$, then

$$0 = \langle x, S(x)\rangle = \langle x, x\rangle + \langle x, T^*T(x)\rangle = \langle x, x\rangle + \langle T(x), T(x)\rangle = \|x\|^2 + \|T(x)\|^2.$$

Hence, $\|x\| = 0 \Rightarrow x = 0$ and $N(S) = \{0\}$. By assignment 3, we have $S$ is injective.

8. (a) Since $T$ is normal, $TT^* = T^*T$. We have $\langle T(x), T(x)\rangle = \langle x, T^*T(x)\rangle = \langle x, TT^*(x)\rangle = \langle T^*(x), T^*(x)\rangle$. Thus $\|T(x)\| = \|T^*(x)\|$.

(b)

$$\begin{aligned}(T - \alpha I)(T - \alpha I)^* &= (T - \alpha I)(T^* - \overline{\alpha} I) = TT^* - \overline{\alpha}T - \alpha T^* + \alpha\overline{\alpha} I \\ &= T^*T - \overline{\alpha}T - \alpha T^* + \alpha\overline{\alpha} I = (T^* - \overline{\alpha} I)(T - \alpha I) = (T - \alpha I)^*(T - \alpha I)\end{aligned}$$

Thus $T - \alpha I$ is normal.

(c) By (a) and (b), $\|(T - \alpha I)(x)\| = \|(T - \alpha I)^*(x)\| = \|(T^* - \overline{\alpha} I)(x)\|$ for any $x \in H$. Hence,

$$T(x) = \alpha x \Leftrightarrow (T - \alpha I)(x) = 0 \Leftrightarrow (T^* - \overline{\alpha} I)(x) \Leftrightarrow T^*(x) = \overline{\alpha} x.$$

(d) Let $T(x) = \lambda_1 x, T(y) = \lambda_2 y$. By (c), $T^*(y) = \overline{\lambda_2} y$. Then

$$\alpha_1\langle x, y\rangle = \langle \alpha_1 x, y\rangle = \langle T(x), y\rangle = \langle x, T^*(y)\rangle = \langle x, \overline{\lambda_2} y\rangle = \lambda_2\langle x, y\rangle.$$

By assumption, $\lambda_1 \neq \lambda_2$. This implies $\langle x, y\rangle = 0$ and $x, y$ are orthogonal.

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