MMAT 5011 Analysis II
2016-17 Term 2
Assignment 4
Suggested Solution

1. **(Parallelogram equality)** Let H be an inner product space, let $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ be the norm induced by the inner product. Then we have the following Parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

Proof.

$$\begin{aligned}\|x+y\|^2 + \|x-y\|^2 =& \langle x+y, x+y\rangle + \langle x-y, x-y\rangle \\ =& (\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle) \\ &+ (\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle) \\ =& 2(\langle x,x\rangle + \langle y,y\rangle) = 2(\|x\|^2 + \|y\|^2).\end{aligned}$$

2. In a real inner product space,

$$\begin{aligned}\frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) =& \frac{1}{4}(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle) \\ =& \frac{1}{4}(\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle) \\ &- \frac{1}{4}(\langle x,x\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle y,y\rangle) \\ =& \langle x,y\rangle\end{aligned}$$

3. (a) $x = y = (1,1), \langle -x, y\rangle = \langle x, y\rangle = 2$, violating $\langle -x, y\rangle = -\langle x, y\rangle$;
   (b) $x = (2,-1), \langle x, x\rangle = -3 < 0$, violating $\langle x, x\rangle \geq 0$;
   (c) $x = y = (1,1), \langle x, y\rangle = 2+2i = \langle y, x\rangle$, violating $\langle x, y\rangle = \overline{\langle y, x\rangle}$.

4. It suffices to show the Parallogram equality does not hold for l^1. Let $x = ((-1/2)^n)_{n=1}^{\infty}, y = (1/2^n)_{n=1}^{\infty}$. Then

$$\|x+y\|_1 = \sum_{i=1}^{\infty} |x_i + y_i| = \frac{2}{3},$$

$$\|x-y\|_1 = \sum_{i=1}^{\infty} |x_i - y_i| = \frac{4}{3}.$$

However

$$\|x+y\|_1^2 + \|x-y\|_1^2 = \frac{20}{9} \neq 2(\|x\|_1^2 + \|y\|_1^2) = 4.$$

5. Consider $y + \alpha z$ for arbitrary vectors $y, z \in X$ and different $\alpha \in \mathbb{C}$. We have

$$0 = \langle T(y+\alpha z), y+\alpha z\rangle = \langle T(y), y\rangle + \overline{\alpha}\langle T(y), z\rangle + \alpha\langle T(z), y\rangle + |\alpha|^2\langle T(z), z\rangle.$$

Thus

$$\overline{\alpha}\langle T(y), z\rangle + \alpha\langle T(z), y\rangle = 0.$$

For real $\alpha \neq 0$, we have $\langle T(y), z\rangle + \langle T(z), y\rangle = 0$; for imaginary α, we have $\langle T(y), z\rangle - \langle T(z), y\rangle = 0$. This implies $\langle T(y), z\rangle = \langle T(z), y\rangle = 0$ for arbitrary y and z. Let $z = T(y)$, then $\langle T(y), T(y)\rangle = 0, T(y) = 0$, thus T is the zero operator.

6. (a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix} + \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$
   (b) Let $M = A+B$, then $M^t = A^t + B^t = A - B$. Thus $A = \frac{1}{2}(M+M^t), B = \frac{1}{2}(M - M^t)$
   (c) It is easy to see that $\dim Sym_{n\times n} = \frac{n^2+n}{2}, \dim Skew_{n\times n} = \frac{n^2-n}{2}, \dim Sym_{n\times n} + \dim Skew_{n\times n} = \frac{n^2+n}{2} + \frac{n^2-n}{2} = n^2 = \dim M_{n\times n}(\mathbb{R})$.

7. $(\Rightarrow)$ Since $x \perp y$, $\langle x, y\rangle = 0$.

$$\begin{aligned}\langle x+\alpha y, x+\alpha y\rangle =& \langle x, x\rangle + \langle x, \alpha y\rangle + \langle \alpha y, x\rangle + \langle \alpha y, \alpha y\rangle \\ =& \langle x, x\rangle + \overline{\alpha}\langle x, y\rangle + \alpha\langle y, x\rangle + \langle \alpha y, \alpha y\rangle \\ =& \langle x, x\rangle + \langle \alpha y, \alpha y\rangle \\ \geq& \langle x, x\rangle\end{aligned}$$

   Thus $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{C}$.

   $(\Leftarrow)$ Since $\|x + \alpha y\| \geq \|x\|$ for all $\alpha \in \mathbb{C}$, we have

$$\langle x+\alpha y, x+\alpha y\rangle \geq \langle x, x\rangle.$$

   Thus

$$|\alpha|^2\langle y, y\rangle + \alpha\langle y, x\rangle + \overline{\alpha}\langle x, y\rangle \geq 0$$

   for all $\alpha \in \mathbb{C}$.
   For $y = 0$, we already have $x \perp y$. Assume that $y \neq 0$. Then let $\alpha = -\frac{\langle x,y\rangle}{\langle y,y\rangle}$, the above inequality gives

$$\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle^2}\langle y, y\rangle - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} - \frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} \geq 0$$

   i.e.,

$$\frac{|\langle x, y\rangle|^2}{\langle y, y\rangle} \leq 0.$$

   Hence $|\langle x, y\rangle| = 0, \langle x, y\rangle = 0, x \perp y$.

8. (a) Let $x, y \in \overline{B}$, for $0 < \lambda < 1$,

$$\|\lambda x + (1-\lambda)y\| \leq \lambda\|x\| + (1-\lambda)\|y\| \leq \lambda + (1-\lambda) = 1.$$

   Thus $\overline{B}$ is convex.
   (b) $\{(x, y) : xy < 0\} = \{(x, y) : x > 0, y < 0\} \cup \{(x, y) : x < 0, y > 0\}$ is the union of the second and fourth quadrant. Since any line segment connecting two points in the second and fourth quadrant respectively must intersects the first or the third quadrant or the origin, the subset is not convex.

9. (a) For any $x \in A, y \in A^\perp$, $\langle x, y\rangle = 0$, thus $x \perp A^\perp, A \subset (A^\perp)^\perp$.

(b) Let $x_n \in A^{\perp}$, $x_n \to x$ in X. Then $\langle x_n, y\rangle = 0$ for all $y \in A$. Letting n tends to infinity, we get $\langle x, y\rangle = 0$ for all $y \in A$. Thus $x \in A^{\perp}$, $A^{\perp}$ is closed in X.

(c) Consider $e_n \in Y, n = 1, 2, 3, \cdots$. Let $x \in Y^{\perp}$, then $\langle x, e_n\rangle = x_n = 0$ for all n. Thus $Y^{\perp} = \{0\}$, $(Y^{\perp})^{\perp} = l^2$.

10. Suppose $x, y \in \mathbb{R}^2$ are adjacent sides of a rhombus. The the diagonals are $x + y$ and $x - y$. Note that for a rhombus, $\|x\| = \|y\|$, we have

$$\langle x + y, x - y\rangle = \langle x, x\rangle - \langle y, y\rangle = 0.$$

This implies the diagonals are perpendicular.

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