MMAT 5011 Analysis II
2016-17 Term 2
Assignment 3
Suggested Solution

1. (a) Let $x, y \in X$ with $T(x) = T(y)$. Since T is linear, $T(x-y) = 0$ and so $x - y \in N(T) = \{0\}$. Thus $x - y = 0$, i.e. $x = y$. This implies that T is injective.

   (b) By part (a), it suffices to show $N(T) = \{0\}$. Let $x \in N(T)$. Then $\|x\| = \|T(x)\| = 0$ and so $x = 0$. This implies $N(T) = 0$.

2. (a) Note that $\|1+2t\|_\infty = \sup_{0\le t\le 1} |1+2t| = 3$.

$$\|T\| = \sup_{\|f\|_1=1} |T(f)| = \sup_{\|f\|_1=1} \left|\int_0^1 f(t)(1+2t)dt\right| \le \sup_{\|f\|_1=1} \|f\|_1\|1+2t\|_\infty = (1)(3) = 3.$$

   (b) Consider $f_n(t) = (n+1)t^n$.

$$\|f_n\|_1 = \int_0^1 (n+1)t^n = 1.$$

$$\|T\| = \sup_{\|f\|_1=1} |T(f)| \ge |T(f_n)| = \left|\int_0^1 (n+1)t^n(1+2t)\,dt\right| = 3 - \frac{2}{n+2}.$$

   Letting n tends to ∞, we have $\|T\| \ge 3$. Together with (a), we have $\|T\| = 3$.

   (c) We claim that $\|S\| = \max_{0\le t\le 1} |g(t)|$.
   Firstly,

$$\|S\| = \sup_{\|f\|_1=1} |S(f)| \le \sup_{\|f\|_1=1} \|f\|_1\|g\|_\infty = \max_{0\le t\le 1} |g(t)|.$$

   To prove the reverse inequality, we construct $f_n(t)$ as follows.
   Let $M = \max_{0\le t\le 1} |g(t)|$. If $M = 0$, $g(t) \equiv 0$ on $[0,1]$, it is obvious $\|S\| = 0$. Without loss of generality, we assume $g(x_0) = M > 0, x_0 \in (0,1)$ (for $g(x_0) = -M$ and the case $x_0 = 0, 1$ we can apply similar procedures as below). For each $n > 0$, by continuity of $g(t)$, there exists $r_n > 0$ such that $g(t) > M - \frac{1}{n}$ for all $t \in (x_0 - r_n, x_0 + r_n) \subset [0,1]$.
   Define

$$f_n(t) = \begin{cases} 0, & t \in [0, x_0 - r_n], \\ \frac{2}{r_n}(t - x_0 + r_n), & t \in (x_0 - r_n, x_0 - \frac{r_n}{2}), \\ 1, & t \in [x_0 - \frac{r_n}{2}, x_0 + \frac{r_n}{2}], \\ 1 - \frac{2}{r_n}(t - x_0 - \frac{r_n}{2}), & t \in (x_0 + \frac{r_n}{2}, x_0 + r_n), \\ 0, & t \in [x_0 + r_n, 1]. \end{cases}$$

   Then f_n is continuous on $[0,1]$ and $\|f_n\|_1 = \frac{3}{2}r_n$. For all large n, $M - \frac{1}{n} > 0$. Then

$$\|S\| = \sup_{\|f_n\|_1\ne 0} \frac{|S(f_n)|}{\|f_n\|_1} \ge \frac{|\int_0^1 f_n(t)g(t)\,dt|}{\|f_n\|_1} = \frac{|\int_{x_0-r_n}^{x_0+r_n} f_n(t)g(t)\,dt|}{\frac{3}{2}r_n}$$

$$\ge \frac{(M - \frac{1}{n})|\int_{x_0-r_n}^{x_0+r_n} f_n(t)\,dt|}{\frac{3}{2}r_n} = M - \frac{1}{n}.$$

   Letting $n \to \infty$, we get $\|S\| \ge M$. Thus completes the proof.

3. T_n is a Cauchy sequence in $B(X,Y)$. For any $\varepsilon > 0$, there exists $N > 0$, such that $\|T_n - T_m\| \leq \varepsilon$, whenever $n, m > N$. Thus

$$|T_n(x) - T_m(x)| \leq \|T_n - T_m\|\|x\| \leq \varepsilon\|x\|.$$

For every fixed x, the right hand side $\varepsilon\|x\|$ can be made arbitrarily small. Hence, $(T_n(x))$ is a Cauchy sequence. Since Y is a Banach space, $T_n(x)$ is a convergent sequence in Y.

4. Let $f(x,y) = ax + by \in (\mathbb{R}^2)'$,

$$\|f\| = \sup_{\|(x,y)\|=1} |ax + by| \leq \sup_{\|(x,y)\|=1} \sqrt{a^2+b^2}\sqrt{x^2+y^2} = \sqrt{a^2+b^2}.$$

The above equality holds when $ay = bx$, thus $\|f\| = \sqrt{a^2+b^2}$. Consequently,

$$\|S(f)\| = \|(a,b)\| = \sqrt{a^2+b^2} = \|f\|.$$

5. Let $y \in V \setminus Z$. Note that every vector x of V can be expressed uniquely as $z + \alpha y$ for some $z \in Z$ and $\alpha \in \mathbb{R}$. Then $f(x) = f(z + \alpha y) = f(z) + \alpha f(y) = \alpha f(y)$. In the same way, $g(x) = \alpha g(y)$. Note that $y \notin Z = N(g)$ and so $g(y) \neq 0$. Let $c = f(y)/g(y)$, then $f(x) = \alpha f(y) = c\alpha g(y) = cg(x)$.

6. Assume on the contrary $x \neq y$. Let $z = x - y \neq 0$. Define a linear functional f on $Z = \operatorname{span}\{z\} = \{az : a \in \mathbb{R}\}$ by $f(az) = a$. By Hahn-Banach Theorem, there exists a linear extension $\tilde{f}$ on X such that $\|f\| = \|\tilde{f}\|$. Since $\tilde{f}(z) = f(z) \neq 0$, we have $\tilde{f}(x) \neq \tilde{f}(y)$, which is a contradiction.

7. (a) Let $x \in c$. $|f(x)| = |\lim_{i\to\infty} x_i| \leq \|x\|_\infty$, thus f is bounded and $\|f\| \leq 1$. Let $x = (1 - \frac{1}{n})_{n=1}^\infty$, then $|f(x)| = |\lim_{i\to\infty} x_i| = 1$, $\|f\| = 1$.

   (b) Take $e_n \in Z$ the n-th unit sequence, then $y_n = T(y)(e_n) = 0$. Thus $y = 0$ and $T(y)$ is the zero linear functional on l^∞.

   (c) By Holder's inequality, $|T(y)(x)| \leq \|y\|_1\|x\|_\infty, \|T(y)\| \leq \|y\|_1$. For ther reverse inequality, let

$$x_i = \operatorname{sign} y_i = \begin{cases} 1, & y_i > 0; \\ 0, & y_i = 0; \\ -1, & y_i < 0. \end{cases}$$

   Then $\|x\|_\infty = 1$, $|T(y)(x)| = \sum_{i=1}^\infty |y_i| = \|y\|_1$. Thus $\|T(y)\| \geq \|y\|_1$.

   (d) Let $y_1, y_2 \in l^1$, $T(y_1) = T(y_2)$. Then $T(y_1 - y_2)(x) = 0$ for all $x \in Z$. By (b), we have $y_1 - y_2 = 0, y_1 = y_2$, T is injective.
   Let f be the linear functional in (a). By Hahn-Banach Theorem, there exists a linear extension g such that $g = f$ on c and $\|g\| = \|f\| = 1$. If there exists some $y \in l^1$ such that $T(y) = g$. Then $y_n = T(y)(e_n) = g(e_n) = f(e_n) = 0$ for all n. Thus $g = 0$ which is a contradiction, T is not surjective.

8. Omitted.

♠ ♡ ♣ ♢ END ♢ ♣ ♡ ♠