MMAT 5011 Analysis II

Review of linear algebra

Defn A vector space V over a field $\mathbb{F}$ is a set with two operations:

addition: $+ : V \times V \longrightarrow V$

scalar multiplication: $\cdot : \mathbb{F} \times V \longrightarrow V$ such that

1. $x+y=y+x$ for all $x,y \in V$
2. $(x+y)+z = x+(y+z)$ $\forall x,y,z \in V$
3. $\exists$ an element $\vec{0} \in V$ such that $x+\vec{0}=x$ $\forall x \in V$
5. For any $x \in V$, $1 \cdot x = x$ $\forall x \in V$
4. For any $x \in V$, $\exists y \in V$ such that $x+y=\vec{0}$

We call $y=-x$

6. $(\alpha\beta)x = \alpha(\beta x)$ $\forall \alpha,\beta \in \mathbb{F}$ and $x \in V$
7. $\alpha(x+y) = \alpha x + \alpha y$ $\alpha \in \mathbb{F}$, $x,y \in V$
8. $(\alpha+\beta)x = \alpha x + \beta x$ $\alpha,\beta \in \mathbb{F}$, $x \in V$

Rmk ① $\mathbb{F}$ is a field, something you can do $+, -, \times, \div$ with nice properties

eg $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}$, $\cancel{\mathbb{Z}}$

Our focus

no division

② Any vector space $/\mathbb{C}$ is also a vector space $/\mathbb{R}$

Example of vector spaces $V/\mathbb{F}$

① $\mathbb{R}^n/\mathbb{R}$ $\mathbb{C}^n/\mathbb{C}$ $\mathbb{C}^n/\mathbb{R}$

dim = n dim = n dim = 2n

$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}$$

$$i\begin{bmatrix}1\\0\\i\end{bmatrix}=\begin{bmatrix}i\\0\\-1\end{bmatrix}$$

② $M_{m\times n}(\mathbb{R})$ = vector space of all m×n matrices with real entries

dim = mn

$$2\times3 \quad \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}+\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}=\begin{bmatrix}2&3&4\\5&6&7\end{bmatrix}$$

$$(-1)\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}=\begin{bmatrix}-1&-2&-3\\-4&-5&-6\end{bmatrix}$$

③ $P_n(\mathbb{F})$ = the vector space of all polynomials with coefficients in $\mathbb{F}$ and degree $\leq n$

$$x^2+ix-7 \in P_4(\mathbb{C})$$

$\dim P_n(\mathbb{F}) = n+1$

$P(\mathbb{F})$ = the vector space of all polynomials with coefficients in $\mathbb{F}$

$\dim P(\mathbb{F}) = \infty$ (countable)

$P(\mathbb{F})$ has basis $\{1, x, x^2, x^3, x^4, \cdots\}$

④ $C[a,b]$ = vector space of all continuous, real-valued functions defined on $[a,b]$

$\dim C[a,b] = \infty$ (uncountable)

Vector subspace

Defn V is a vector space / $\mathbb{F}$

A subset $W \subseteq V$ is called a vector subspace if

① $\vec{0} \in W$

② $x+y \in W$ if $x, y \in W$ ↙ closed under addition

③ $\alpha x \in W$ if $\alpha \in \mathbb{F}$, $x \in W$

↖ closed under scalar multiplication

Rmk A vector subspace is a vector space itself with same addition and scalar multiplication

eg1 $\{(x,y,0) : x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$

is a vector subspace of $\mathbb{R}^3$

eg2 $P(\mathbb{R}) \subseteq P(\mathbb{C}) \supseteq P_n(\mathbb{C})$

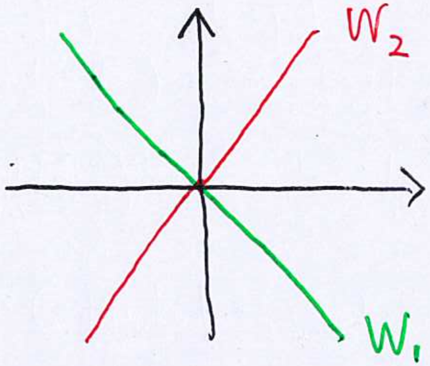
↗ vector subspace over $\mathbb{R}$

↑ vector subspace over $\mathbb{R}$ or $\mathbb{C}$

eg3 Let $V = \mathbb{R}^2$

$W_1 = \{(x,y) : x+y=0\}$

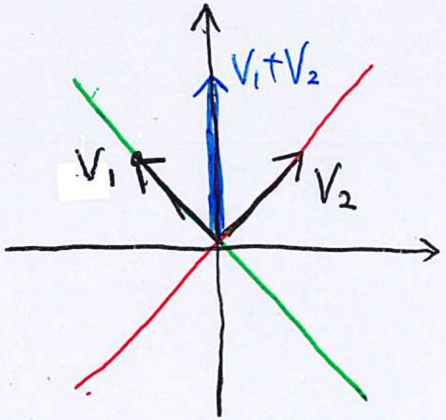
$W_2 = \{(x,y) : x-y=0\}$



$W_1, W_2, \overset{\{0\}}{\overset{\parallel}{W_1 \cap W_2}}$ are vector subspace of V

Q Is $W_1 \cup W_2$ a vector subspace of V?

Ans No



$V_1 = (-1,1) \in W_1$

$V_2 = (1,1) \in W_2$

$\Rightarrow V_1, V_2 \in W_1 \cup W_2$

$0+2 \neq 0$

$V_1 + V_2 = (0,2) \notin W_1, W_2$

$0-2 \neq 0$

$\Rightarrow V_1 + V_2 \notin W_1 \cup W_2$

$W_1 \cup W_2$ fails ② in defn of vector subspace

$\Rightarrow W_1 \cup W_2$ is not a vector subspace

Q Let W_1, W_2 are vector subspaces of V

Show that $W_1 \cap W_2$ is a vector subspace of V

Sol

①. W_1 is a vector subspace of $V \Rightarrow \vec{0} \in W_1$

W_2 is " " " " " $\Rightarrow \vec{0} \in W_2$

$\Rightarrow 0 \in W_1 \cap W_2$

② Let $x, y \in W_1 \cap W_2$

$\therefore x, y \in W_1$

W_1 is a vector subspace $\Rightarrow x+y \in W_1$

Similarly, $x+y \in W_2$

$\Rightarrow x+y \in W_1 \cap W_2$

③ Let $x \in W_1 \cap W_2$, $\alpha \in \mathbb{F}$

Then $x \in W_1$

W_1 is a vector subspace

$\Rightarrow \alpha x \in W_1$

Similarly $\alpha x \in W_2$

$\Rightarrow \alpha x \in W_1 \cap W_2$

We proved $W_1 \cap W_2$ satisfies

① , ② , ③

∴ $W_1 \cap W_2$ is a vector subspace of V

Basis and dimension

Def Let V be a vector space over $\mathbb{F}$, $S \subseteq V$ be a subset

① $\text{Span}\, S \triangleq \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : n \geq 0,\ x_1, x_2, \cdots, x_n \in S,\ \alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}\}$

= the set of all linear combinations of S

Note that Span S is a vector subspace of V.

② S is linearly dependent if there exist pairwisely distinct $x_1, x_2, \cdots, x_n \in S$ and $a_1, a_2, \cdots, a_n \in \mathbb{F}$, not all zero

Such that $a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \vec{0}$

S is said to be linearly independent if S is not linearly dependent

③ S is said to be a basis (Hamel basis) of V if

S is linearly independent and

$\operatorname{Span} S = V$

Most important feature of a basis:

If S is a basis of V, then any $x \in V$ can be expressed as

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n$$

with $x_1, x_2, \cdots, x_n \in S$, pairwisely distinct and $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathbb{F}$

in a unique way

$\operatorname{Span} S = V \Rightarrow$ can, l.n. indept $\Rightarrow$ unique

eg 1

is a basis

Let $V = \mathbb{R}^3$, $S = \{e_1, e_2, e_3\}$, $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 1 \\ -7 \end{bmatrix} = (3)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + (1)\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + (-7)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

or any other vector in $\mathbb{R}^3$

unique, no other choices

Another basis $= \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3000 \end{bmatrix} \right\}$

In general, standard basis of $\mathbb{R}^n$ is

$\{e_1, e_2, \cdots, e_n\}$

eg 2

$V = P_4(\mathbb{C})$ has a basis

$\{1, x, x^2, x^3, x^4\} / \mathbb{C}$

Another basis is

$\{1+i, x+2, ix^2+x, x^3, (1+i)x^4\}$

Q How about if we regard V as a vector space / $\mathbb{R}$?

Ans

$\{1, i, x, ix, x^2, ix^2, x^3, ix^3, x^4, ix^4\}$

is a basis for $V/\mathbb{R}$

Thm Every vector space $V/\mathbb{F}$ has a basis

(Pf by set theory)

Thm If S, S' are both bases of V, then S and S' have same cardinality

If S and S' are finite, it just means that they have the same number of elements

This cardinality is defined to be dim V (dimension)

Thm Let $\dim V = n$, $S \subseteq V$ be a subset with n elements.

Then S is linearly independent $\Leftrightarrow$ Span $S = V$

In particular, S is a basis in this situation.

e.g. Are they bases of $P_2(\mathbb{R})$?

① $\{x^2+2x+2,\ x^2+1,\ x^2+4x+4\} = S_1$

② $\{x^2+2x+2,\ x^2+1,\ x^2+4x+3\} = S_2$

Sol ① Check linear independence:

Suppose

$$a_1(x^2+2x+2) + a_2(x^2+1) + a_3(x^2+4x+4) \equiv 0$$

then $(a_1+a_2+a_3)x^2 + (2a_1+4a_3)x + (2a_1+a_2+4a_3) \equiv 0$

$$\Rightarrow \begin{cases} a_1 + a_2 + a_3 = 0 \\ 2a_1 \qquad\quad + 4a_3 = 0 \\ 2a_1 + a_2 + 4a_3 = 0 \end{cases}$$

Gaussian elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\therefore \begin{array}{r} a_3 = 0 \\ a_2 - a_3 = 0 \\ a_1 + a_2 + a_3 = 0 \end{array} \begin{array}{c} \Rightarrow \\ \Rightarrow \\ \Rightarrow \end{array} \boxed{\begin{array}{c} a_3 = 0 \\ a_2 = 0 \\ a_1 = 0 \end{array}}$$

∴ S_1 is linearly independent

Note $\dim P_2(\mathbb{R}) = 3$, S_1 has 3 elements

$\Rightarrow$ Span $S_1 = P_2(\mathbb{R})$

$\Rightarrow$ S_1 is a basis of $P_2(\mathbb{R})$ ▨

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② Suppose $a_1(x^2+2x+2) + a_2(x^2+1) + a_3(x^2+4x+3) \equiv 0$

Similarly $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 0 & 4 & 0 \\ 2 & 1 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

pivot

a_1, a_2 are lead/pivot/basic variables

a_3 is a free variable

$\boxed{1}$ leading 1

The system is consistent and has infinitely many solutions. In particular, $a_3 = 1$, $a_2 = 1$, $a_1 = -2$ is one of them $\Rightarrow$ S_2 is not lin. indept $\Rightarrow$ not a basis

Something about Matrices $\left(\begin{matrix}\text{Most of the time}\\ \mathbb{F}=\mathbb{C} \text{ or } \mathbb{R}\end{matrix}\right)$

- Matrix multiplication

$A \in M_{m\times n}(\mathbb{F})$ $\quad B \in M_{n\times k}(\mathbb{F})$

↑

m rows, n columns $\quad AB \in M_{m\times k}(\mathbb{F})$

- Gaussian elimination
  - Augmented matrix
  - elementary row operations (3 types)
  - (Reduced) row echelon form

leading 1's

$$\begin{bmatrix} \boxed{1} & 2 & 3 \\ \enclose{circle}{\boxed{1}} & 4 & 5 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & \enclose{circle}{2} & 3 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

ref? × ✓ ✓

rref? × × ✓

- lead/pivot/basic variables, free variables

$$\left[\begin{array}{ccccc|c} \boxed{1} & 2 & 3 & 4 & 5 & 9 \\ 0 & 0 & \boxed{1} & 6 & 7 & 10 \\ 0 & 0 & 0 & \boxed{1} & 8 & 11 \end{array}\right]$$

x_1, x_3, x_4 pivot

x_2, x_5 free

- number of solutions

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ \enclose{roundedbox}{0 \quad 0 \quad 0} & & & \enclose{roundedbox}{3} \end{array}\right] \qquad \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 1 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & \boxed{1} & 3 \end{array}\right] \qquad ⊛ \left[\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 1 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

no solution

unique solution

no free variable

infinitely many solutions

z is free

- Express solutions in parametric form

eg. ⊛ $(x,y,z) = (-1,2,0) + t(-3,1,1)$

- Finding inverse of a square matrix

$$\left[\underbrace{\begin{matrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 2 & 5 & 7 \end{matrix}}_{A} \middle| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}\right] \sim \left[\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \middle| \underbrace{\begin{matrix} -1 & -1 & 1 \\ -8 & -1 & 4 \\ 6 & 1 & -3 \end{matrix}}_{A^{-1}}\right]$$

The followings are equivalent

- For $A \in M_{n\times n}(\mathbb{F})$, TFAE

① A is invertible ② $\mathrm{rref}(A) = I_n$

③ Each diagonal entry of any ref of A is non-zero

④ $A\vec{x} = \vec{b}$ has unique solution for any $\vec{b} \in \mathbb{F}^n$

⑤ $\mathrm{rank}\, A = n$ ⑥ $\mathrm{nullity}\, A = 0$

⑦ $\det A \neq 0$

Linear transformations

Def let V, W be vector space $/\mathbb{F}$.

A map $T: V \to W$ is called a linear transformation if $\forall x, y \in V$, $a \in \mathbb{F}$

- $T(x+y) = T(x) + T(y)$
- $T(ax) = aT(x)$.

eg $A \in M_{m \times n}(\mathbb{F})$. Then $T: \mathbb{F}^n \to \mathbb{F}^m$ defined by $T(x) = Ax$ is linear

Properties of a linear transformation $T: V \to W$

1. $T(\underset{\text{in } V}{\underset{\uparrow}{\vec{0}}}) = \underset{\text{in } W}{\underset{\uparrow}{\vec{0}}}$

2. $T\left(\sum_{i=1}^{k} a_i x_i\right) = \sum_{i=1}^{k} a_i T(x_i)$

for any $x_1, x_2, \cdots, x_k \in V$, $a_1, a_2, \cdots, a_k \in \mathbb{F}$

Def let $T: V \to W$ be a linear transformation

Define

null space (kernel) $N(T) = \{x \in V \mid T(x) = \vec{0}\}$

range (image) $R(T) = \{T(x) \mid x \in V\}$

Note $N(T) \subseteq V$, $R(T) \subseteq W$ are linear subspaces.
↑ i.e vector subspaces

Define

$$\text{nullity}(T) = \dim N(T)$$
$$\text{rank}(T) = \dim R(T)$$

Properties For $T: V \to W$

T is one-to-one $\iff N(T) = \{0\}$

$\iff \text{nullity}(T) = 0$

T is onto $\iff R(T) = W$

$\iff \text{rank}(T) = \dim W$ (if $\dim W < \infty$)

Dimension theorem: If $\dim V < \infty$

$$\boxed{\text{nullity}(T) + \text{rank}(T) = \dim V}$$

Normed Space ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

Defn Let X be a vector space $/\mathbb{F}$

A norm on X is a function

$$\|\cdot\| : X \longrightarrow \mathbb{R}$$

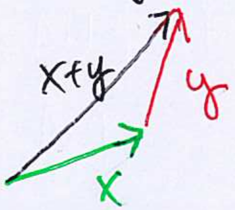
such that $\forall x, y \in X$, $\alpha \in \mathbb{F}$

N1. $\|x\| \geq 0$

N2. $\|x\| = 0 \iff x = \vec{0}$

N3. $\|\alpha x\| = |\alpha| \, \|x\|$ $\quad |\alpha| =$ absolute value if $\mathbb{F} = \mathbb{R}$, modulus if $\mathbb{F} = \mathbb{C}$

N4. $\|x + y\| \leq \|x\| + \|y\|$ Triangle inequality



$(X, \|\cdot\|)$ is called a normed space

or simply X

Proposition A normed space X is also a metric space with metric

$$d(x,y) = \|x - y\|$$

Pf Check the conditions for metric space

M1: $d(x,y) = \|x - y\| \geq 0$ by N1

M2:
$$\begin{aligned} d(x,y) = 0 &\iff \|x - y\| = 0 \\ &\iff x - y = \vec{0} \text{ by N2} \\ &\iff x = y \end{aligned}$$

M3:
$$\begin{aligned} d(x,y) &= \|x - y\| \\ &= \|(-1)(y - x)\| \\ &= |-1| \, \|y - x\| \quad \text{by N3} \\ &= \|y - x\| = d(y,x) \end{aligned}$$

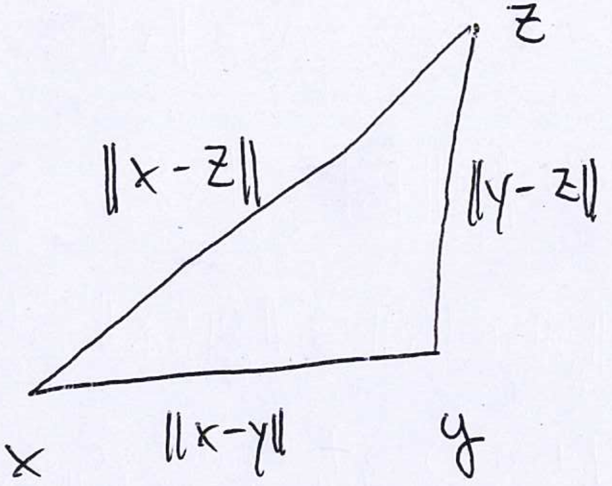
M4: $d(x,y) + d(y,z)$

$= \|x-y\| + \|y-z\|$

$\geq \|(x-y)+(y-z)\|$ by N4

$= \|x-z\|$

$= d(x,z)$



Rmk If X is a normed space, then

$\|\cdot\| : X \to \mathbb{R}$ is a continuous function

Example of normed space

Rmk $\sqrt{\alpha^2} = |\alpha|$

① $\mathbb{R}^n$, with norm

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$\mathbb{C}^n$, with norm

$$\|z\| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2}$$

$z_1 \overline{z_1} = |z_1|^2$

N1, N2, N3 Easy to verify

N4 a little bit harder

eg 2 C[0,1] with norm defined by

$$\|f\| = \max_{0\le x\le 1} |f(x)| \qquad (\mathbb{F}=\mathbb{R})$$

Note $\|f\|$ is well-defined because of

Extreme value theorem,

If $g(x)$ is continuous on a closed and bounded interval, then $g(x)$ has maximum and minimum

Verify that $\|\cdot\|$ is a norm

Sol: Check

N1: For any $f \in C[0,1]$, $x \in [0,1]$

$$|f(x)| \ge 0 \Rightarrow \|f\| = \max_{0\le x\le 1} |f(x)| \ge 0$$

N2:

$$\|f\| = 0 \Leftrightarrow \boxed{\max_{0\le x\le 1} |f(x)| = 0}$$

$$\Leftrightarrow 0 \le |f(x)| \le 0 \text{ for any } 0\le x\le 1$$

$$\Leftrightarrow f(x) = 0 \text{ for any } 0\le x\le 1$$

$$\Leftrightarrow f = \vec{0}, \text{ the zero fuction}$$

N3:

$$\begin{aligned}\|\alpha f\| &= \max_{0\le x\le 1} |(\alpha f)(x)| \\ &= \max_{0\le x\le 1} |\alpha(f(x))| \\ &= \max_{0\le x\le 1} |\alpha|\,|f(x)| \\ &= |\alpha| \max_{0\le x\le 1} |f(x)| \\ &= |\alpha|\,\|f\|\end{aligned}$$

N4: Let $f, g \in C[0,1]$

Suppose $x \in [0,1]$

$$|(f+g)(x)| = |f(x)+g(x)|$$

triangle inequality for real numbers $\leq |f(x)| + |g(x)|$

$$\leq \|f\| + \|g\|$$

$$\Rightarrow \max_{0 \leq x \leq 1} |(f+g)(x)| \leq \|f\| + \|g\|$$

$$\Rightarrow \|f+g\| \leq \|f\| + \|g\|$$

N1, N2, N3, N4 ✓

$\Rightarrow \|f\| = \max_{0 \leq x \leq 1} \|f(x)\|$ is a norm

Q Let $C(\mathbb{R})$ be the vector space of all continuous function on $\mathbb{R}$

Does $\|f\| = \max_{x \in \mathbb{R}} |f(x)|$ define a norm on $C(\mathbb{R})$?

A No: Problem is $\|f\|$ may not be well-defined

$f(x) = x \ \forall x \in \mathbb{R}$, $|f(x)|$ has no maximum

Q How about on $C(0,1)$ = vector space of all continuous function on $(0,1)$

A No: Same problem:

eg. $f(x) = \frac{1}{x}$ $|f(x)|$ is unbounded

Modification:

eg Let $A \subseteq \mathbb{R}$ be a subset

$B(A)$ = the vector space of all bounded functions defined on A

$$= \{ f : A \to \mathbb{R} : \exists M > 0 \text{ such that } |f(x)| \le M \ \forall x \in A \}$$

Then

$$\|f\| = \sup_{x \in A} |f(x)|$$

defines a norm on $B(A)$

Rmk $\max_{x \in A} |f(x)|$ may not exist: eg. $f(x) = 1 - \frac{1}{x}$

$A = [1, \infty)$