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The reducibility of surgered 3-manifolds

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Abstract

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Suppose M is an irreducible 3-manifold with torus T as a boundary component. We will show that if there are two different Dehn fillings along T such that the resulting manifolds are both reducible, then the distance between the filling slopes is at most two.

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The reducibility of surgered manifolds has attracted much attention in the past years (see for example [2, 3, 4, 5]). Let M be a connected orientable irreducible 3-manifold with torus T as a boundary component. Suppose that γ_1, γ_2 are two slopes on T with geometric intersection number $\Delta = \Delta(\gamma_1, \gamma_2)$. Let $M(\gamma_i)$ be the manifold obtained by gluing a solid torus J_i to M so that the boundary of a meridian disc has slope γ_i. An interesting unsolved problem is the *reducibility conjecture*, which says that if both $M(\gamma_1)$ and $M(\gamma_2)$ are reducible, then $\Delta \leqslant 1$. There is some strong evidence for this conjecture. For example, Gordon and Luecke observed that if $\Delta \geqslant 2$, then both $M(\gamma_1)$ and $M(\gamma_2)$ will be the connected sum of two lens spaces. Especially, the conjecture is true if either M is noncompact or it has more than one boundary component. In the general case, Gordon and Litherland [3] proved that Δ cannot be greater than 4. In this paper we will prove

Theorem 0.1. *If $M(\gamma_1)$ and $M(\gamma_2)$ are both reducible, then $\Delta \leqslant 2$.*

This rules out the possibility of $\Delta=3$, 4. A nontrivial example of $\Delta=1$ was presented in [3]. (Surgery on cable knots in reducible manifolds gives "trivial" examples.) The possibility of $\Delta=2$ remains a challenging open problem.

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1. Notations and definitions

Suppose $M(\gamma_1)$ is reducible. Let S_i be a reducing sphere of $M(\gamma_i)$. Isotope S_1 so that $S_1 \cap J_1 = v_1 \cup \cdots \cup v_{n_1}$ is a disjoint union of meridian discs, where v_i is subscripted so that $v_1, \ldots, v_{n_1}$ appear consecutively in J_1. Similarly, we have $S_2 \cap J_2 = w_1 \cup \cdots \cup w_{n_2}$. We choose S_i so that n_i is minimal. By an isotopy of S_2, we may assume that ∂v_i intersects ∂w_j at Δ points for all i, j. Thus, when we travel around ∂v_i, we will consecutively meet $\partial w_1, \partial w_2, \ldots, \partial w_{n_2}, \ldots, \partial w_1, \ldots, \partial w_{n_2}$ (repeated Δ times).

Let $P_i = S_i \cap M$. It is a planar surface with n_i boundary components. Since n_i is minimized, P_i is incompressible and boundary incompressible. By a further isotopy of S_2 (fixing $\bigcup w_i$), we may assume that P_1 and P_2 are in general position, and $P_1 \cap P_2$ has the minimal number of components. A standard innermost disc argument then guarantees that no circle component of $P_1 \cap P_2$ bounds a disc in either P_1 or P_2. Since P_i is boundary incompressible, no arc in $P_1 \cap P_2$ can be boundary parallel in P_i.

Let $\Gamma_1 = (\bigcup v_i) \cup \{\text{arc components of } P_1 \cap P_2\}$. Γ_1 is considered a graph in S_1: It has the discs v_i as its "fat" vertices, and the arcs in $P_1 \cap P_2$ as its edges. There are Δn_2 edges incident to each vertex v_i of Γ_1. If e is such an edge, and an end of e is in $\partial v_i \cap \partial w_j$, then we give this end of e the label j. In this way, each end of each edge in Γ_1 has a label. When we travel around ∂v_i in some direction, the labels appear in the order $1, 2, \ldots, n_2, \ldots, 1, \ldots, n_2$ (repeated Δ times). The labels are considered to be a mod n_2 number. Thus, for example, n_2+1 is the same label as 1.

In the same way we can define $\Gamma_2 = (\bigcup w_j) \cup \{\text{arc components of } P_1 \cap P_2\}$ as a graph in S_2, and we label the ends of edges of Γ_2 in a similar way. Since the edges in Γ_i are arcs in $P_1 \cap P_2$, each edge e in Γ_1 can also be considered as an edge in Γ_2. Note that if in Γ_1 an edge e is incident to v_i and has label j at that end, then in Γ_2 it is incident to w_j and has label i at that end.

Given an orientation to S_i and K_i (the central curve of J_i), we can refer to + or − vertex, according to the sign of its intersection with K_i. Two vertices are *parallel* if they have the same sign. Otherwise they are *antiparallel*. Since M is orientable, we have the following

Parity rule: *An edge e connects parallel vertices in Γ_1 if and only if it connects antiparallel vertices in Γ_2.*

A pair of edges $\{e_1, e_2\}$ in Γ_i is called an *S-cycle* if it is a Scharlemann cycle of length 2. That is, e_1, e_2 are adjacent parallel edges connecting a pair of parallel vertices in Γ_i, and have the same two labels at their ends. Note that in this case the two labels are successive, and we call them the labels of the S-cycle.

A set of four parallel edges $\{e_1, e_2, e_3, e_4\}$ is called an *extended S-cycle* if $\{e_2, e_3\}$ is an S-cycle, and e_i is adjacent to e_{i+1}, $i = 1, 2, 3$. (I.e., e_i and e_{i+1} are parallel, and there are no edges between them.)

2. Proof of Theorem 0.1

By [3], Theorem 0.1 is true if n_1 or n_2 is less than 4. So we assume that $n_i \geqslant 4$. Suppose that $\{e_1, e_2\}$ is an S-cycle in Γ_2 with labels $\{r, r+1\}$. Then on the other graph Γ_1, the two edges e_1, e_2 connect the vertex v_r to v_{r+1}.

Lemma 2.1. *If there is a disc B in S_1 such that $e_1 \cup e_2 \cup v_r \cup v_{r+1} \subset B$, then*

$$|B \cap J_1| \geqslant (n_1/2) + 1.$$

Proof. Let V be the part of J_1 between v_r and v_{r+1}. Then a regular neighborhood N of $V \cup B$ is a solid torus. Let D be the disc in P_2 bounded by e_1, e_2 and two arcs α, β on ∂P_2. Then in $M(\gamma_1)$, the curve ∂D is contained in $V \cup B$, and intersects a meridian disc of N twice in the same direction. So a regular neighborhood of $V \cup B \cup D$ is a projective space P (see [1, p. 280] for details). If $B \cap \Gamma_1$ has k vertices, then $P \cap J_1$ has $k-1$ components (since the vertices v_r and v_{r+1} are connected by V). Thus, ∂P is a reducing sphere, and $\partial P \cap J_1$ has $2(k-1)$ components. By the minimality of n_1, $2(k-1) \geqslant n_1$. Therefore $k \geqslant (n_1/2)+1$. □

Lemma 2.2. *If Γ_2 has two S-cycles $\{e_1, e_2\}$ and $\{e_1', e_2'\}$ with labels $\{r, r+1\}$ and $\{s, s+1\}$ respectively, then $\{r, r+1\} = \{s, s+1\}$.*

Proof. If $\{r, r+1\} \neq \{s, s+1\}$, then on Γ_1, the complement of $(e_1 \cup e_2 \cup v_r \cup v_{r+1}) \cup (e_1' \cup e_2' \cup v_s \cup v_{s+1})$ has three components D_1, D_2 and A such that $\partial \bar{D}_1 \subset e_1 \cup e_2 \cup v_r \cup v_{r+1}$, and $\partial \bar{D}_2 \subset e_1' \cup e_2' \cup v_s \cup v_{s+1}$, where $\bar{D}_i$ is the closure of D_i. (A is an annulus or a disc, depending on whether $\{r, r+1\} \cap \{s, s+1\} = \emptyset$.) Since $D_1 \cup D_2$ does not contain $\{v_r, v_{r+1}, v_s, v_{s+1}\}$, it contains at most $n_1 - 3$ vertices of Γ_1. Thus, one of the D_i, say D_1, contains at most $(n_1-3)/2$ vertices of Γ_1. Let $D = D_1 \cup e_1 \cup e_2 \cup v_r \cup v_{r+1}$. Then $|D \cap J_1| \leqslant (n_1+1)/2$, contradicting Lemma 2.1. □

Lemma 2.3. *Γ_2 has no extended S-cycles.*

Proof. Let $\{e_1, e_2, e_3, e_4\}$ be an extented S-cycle on Γ_2 such that $\{e_2, e_3\}$ has labels $\{r, r+1\}$. Then, since the labels on ∂w_i are successive, both e_1 and e_4 have labels $r-1$ and $r+2$ on their ends. Thus, on Γ_1, e_2, e_3 connect v_r to v_{r+1}, and e_1, e_4

connect v_{r-1} to v_{r+2}. Since we have assumed that $n_i \geqslant 4$, the two sets $C_1 = e_1 \cup e_4 \cup v_{r-1} \cup v_{r+2}$ and $C_2 = e_2 \cup e_3 \cup v_r \cup v_{r+1}$ are disjoint in S_1. So we can choose disjoint discs B_1, B_2 on S_1 such that $B_i \supset C_i$, and B_i has boundary disjoint from the vertices of Γ_1. By Lemma 2.1, $|B_2 \cap J_1| > n_1/2$. Hence $|B_1 \cap J_1| < n_1/2$. The rest of the proof is similar to that of Lemma 2.1: Choose V to be the part of J_1 which is between v_{r-1} and v_{r+2}, and contains v_r and v_{r+1}. Let D be the disc on P_2 bounded by e_1, e_4 and two other arcs α and β on ∂P_2. Then $\partial D \subset \partial V \cup B_1$. Note that Int $D \cap P_1 \subset e_2 \cup e_3 \subset B_2$. Thus, Int $D \cap (V \cup B_1) = \emptyset$. Now it is easy to see that a regular neighborhood N of $B_1 \cup V \cup D$ is a projective space, and $|\partial N \cap J_1| < n_1$, contradicting the minimality of n_1. □

Lemma 2.4. *Γ_2 cannot have more than $(n_1/2)+1$ parallel edges connecting a pair of parallel vertices.*

Proof. Suppose $e_1, \ldots, e_t$ are parallel edges connecting w_a to w_b, where $t = n_1/2+2$ if n_1 is even, and $t = (n_1-1)/2+2$ if n_1 is odd. Then there is an S-cycle within them (see [1, Corollary 2.6.7]). Suppose that $\{e_i, e_{i+1}\}$ is an S-cycle. If $i \neq 1$, $t-1$, then $\{e_{i-1}, e_i, e_{i+1}, e_{i+2}\}$ would be an extended S-cycle, contradicting Lemma 2.3.

Now suppose $i=1$. By relabeling v_i if necessary, we may assume that e_i has label i at w_a. Since $\{e_1, e_2\}$ form an S-cycle, e_1 has label 2 at w_b, e_2 has label 1 at w_b. Thus, e_i has label $n_1 - i + 3$ at w_b for $i \geqslant 3$. If n_1 were odd, e_t would have label t at both ends, contradicting the parity rule. If n_1 were even, e_{t-1} would have label $n_1 - ((t-1)-3) = t$ at w_b, and e_t would have label $t-1$ at w_b. Thus, $\{e_{t-1}, e_t\}$ would be an S-cycle with labels $\{t-1, t\}$. So we have two S-cycles with different set of labels, contradicting Lemma 2.2. The proof of the case $i = t-1$ is similar. □

Lemma 2.5. *Suppose that e', e'' are two parallel edges connecting a pair of parallel vertices. If they have a label r in common, then they form an S-cycle.*

Proof. Let $e' = e_1, e_2, \ldots, e_k = e''$ be the successive parallel edges between e' and e''. The edges e', e'' cannot both have label r at the same vertex, otherwise $k \geqslant n_1+1$, contradicting Lemma 2.4. So suppose e' (respectively e'') has label r at w_a (respectively w_b). We assume that e_2 has label $r+1$ at w_a. (The other case is similar.) Then e_i has label $r+i-1$ at w_a. Since the vertices are parallel, the label of e_{k-i} at w_b is $r+i$. Now k must be even, otherwise the edge $e_{(k+1)/2}$ has the same label $r+(k-1)/2$ at both ends, contradicting the parity rule. Let $t = k/2$. Then $\{e_t, e_{t+1}\}$ is an S-cycle, because the two edges are adjacent and both have labels $\{r+t-1, r+t\}$ at their ends. If $k > 2$, we would have an extended S-cycle $\{e_{t-1}, e_t, e_{t+1}, e_{t+2}\}$, contradicting Lemma 2.3. Therefore, $k=2$, and $\{e', e''\}$ is an S-cycle. □

Lemma 2.6. *One of Γ_1 and Γ_2 satisfies:*

Each vertex is incident to an edge connecting it to an antiparallel vertex. $(*)$

Proof. If Γ_1 does not have property $(*)$, then there is a vertex v_r such that each edge incident to it will connect it to a parallel vertex. By the parity rule, for each vertex w_i of Γ_2, all the edges incident to w_i with label r will connect w_i to antiparallel vertices. Thus, Γ_2 has property $(*)$. □

Now we suppose that Γ_2 has property $(*)$. Let Γ_2' be the subgraph of Γ_2 consisting of edges connecting parallel vertices. A component F' of Γ_2' is called an extremal component if there is a disc D such that $D \cap \Gamma_2' = F'$. In this case $F = D \cap \Gamma_2$ is a graph in D. If e is an edge in Γ_2 connecting a vertex of F' to an antiparallel vertex, then $e \cap D$ is an edge of F connecting that vertex to ∂D. Such an edge is called a boundary edge of F. Property $(*)$ means that each vertex of F belongs to a boundary edge.

The reduced graph $\bar{F}$ of F is defined to be the graph obtained from F by choosing one edge from each family of parallel edges. Define the *valency* of a vertex to be the number of edges incident to it.

Lemma 2.7. *Let Γ be a graph in a disk D with no trivial loops or parallel edges, such that every vertex of Γ belongs to a boundary edge. Then either Γ has only one vertex, or there are at least two vertices of valency at most* 3, *each of which belongs to a single boundary edge.*

This follows immediately from the proof of [1, Lemma 2.6.5]. In particular, it is true for the graph $\bar{F}$ in D.

Lemma 2.8. *Suppose $\Delta \geqslant 3$. If w_r is a vertex of $\bar{F}$ which has valency at most* 3, *then Γ_1 has an S-cycle with r as one of its labels.*

Proof. The hypothesis implies that in Γ_2, there are at most two families of parallel edges connecting w_r to parallel vertices, and if there are two, then they are successive. By Lemma 2.5, each family has at most $(n_1/2)+1$ edges. So there are at most n_1+2 successive edges connecting w_r to parallel vertices, and all the others connect w_r to antiparallel vertices. On Γ_1 it means:

> *Except for at most two vertices, each vertex v_i is incident to at most one edge that has label r at v_i and connects v_i to an antiparallel vertex. For each exceptional vertex, there are at most two such edges.* $(**)$

Denote by K the subgraph of Γ_1 consisting of edges which connects parallel vertices and has one end labeled r. Let E be an extremal component of K and let D be a disc in S_1 such that $D \cap K = E$. Let E_1 be the subgraph of Γ_1 consisting of edges which has one end at a vertex of E and has label r at that vertex. Restricting this graph to D, we get a graph $H = E_1 \cap D$ in D. The disc D can be chosen so that no edges of H have both ends in ∂D. The interior edges are just the edges of E, while a boundary edge corresponds to an edge in Γ_1 which connects a vertex of E

to an antiparallel vertex, and has label r at the end in E. By (**) above, at most two vertices of H have two boundary edges. We want to show that H has a pair of parallel inner edges.

To prove this, we shrink the boundary of D into one point. Then H becomes a connected graph H' in a sphere S^2. Note that among all the edges connecting to a certain vertex of H, just Δ of them have label r at that vertex. Thus, if H has v vertices, then H' has $v+1$ vertices and Δv edges. Denote by f_1 the number of faces bounded by two edges, and by f_2 the number of the other faces (which must be bounded by at least three edges because H' has no trivial loops). Then $2f_1+3f_2 \leqslant$ 2 (number of edges) $=2\Delta v$. Thus, $f_2 \leqslant (2\Delta v - 2f_1)/3$. So we have

$$\begin{aligned} 2 = \chi(S^2) &= (v+1) - (\Delta v) + (f_1 + f_2) \\ &\leqslant 1 + (1 - \Delta/3)v + f_1/3 \leqslant 1 + f_1/3. \end{aligned}$$

It follows that $f_1 \geqslant 3$. Since H has at most two pairs of parallel boundary edges, it must have at least one pair of parallel interior edges. The conclusion now follows from Lemma 2.5. □

Proof of Theorem 0.1. Suppose $\Delta \geqslant 3$. By Lemma 2.8, for each extremal component F' of Γ_2', and each vertex w_r of valency at most 3 in the corresponding graph $\bar{F}$, there is an S-cycle in Γ_1 with r as a label. By property (*), Γ_2' is disconnected. So there are at least two extremal components. By Lemma 2.7, if F' has more than one vertex, then there are two vertices of valency at most 3 in $\bar{F}$. Therefore, if either Γ_2' has more than two extremal components, or one of the extremal components has more than one vertex, then we can find three different labels, each of which is a label of some S-cycles in Γ_1. But since an S-cycle has only two labels, this would imply that there exist two S-cycles with different pair of labels, contradicting Lemma 2.2. So we suppose that $\{w_r\}$, $\{w_s\}$ are the only extremal components of Γ_2'. Since we have assumed $n_i \geqslant 4$, there must be some other components in Γ_2'. Each of these components will separate $\{w_r\}$ from $\{w_s\}$, for otherwise we could find another extremal component. It follows that in Γ_2, there is no arc connecting w_r to w_s. Now suppose $\{e_1, e_2\}$ (respectively $\{e_1', e_2'\}$) is an S-cycle in Γ_1 with r (respectively s) as a label. Since e_i cannot connect w_r to w_s in Γ_2, we conclude that s is not a label of e_i. Thus, these two S-cycles have different sets of labels, again a contradiction to Lemma 2.2. Therefore we have $\Delta \leqslant 2$, and the theorem follows. □

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