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Dehn Surgery and 3-Manifolds

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Introduction

These notes are somewhat expanded versions of the six lectures given at the 2006 Park City Mathematics Institute Graduate Summer School. The main focus of the lectures was exceptional Dehn surgeries on knots, and, more generally, exceptional Dehn fillings on hyperbolic 3-manifolds.

In Lecture 1 we describe the crude classification of 3-manifolds that comes from cutting them along essential surfaces of non-negative Euler characteristic, and say what this means for exteriors of knots. In Lecture 2 we discuss Dehn surgery on knots, and in particular describe a construction, framed surgery on knots on surfaces, which is the source of many examples of exceptional Dehn surgeries. Lecture 3 summarizes some facts and conjectures about exceptional Dehn surgeries on knots. In Lecture 4 we introduce rational tangle fillings on tangles; these induce Dehn fillings on the double branched cover of the tangle. Tangle fillings have the advantage that they are easy to visualize, and although they impose a symmetry on the manifold in question, nevertheless it turns out that many examples of exceptional Dehn fillings arise in this way. Lecture 5 gives more examples of exceptional Dehn fillings derived from tangles. In Lecture 6 we discuss some classification results about exceptional Dehn fillings; many of these take the form that a hyperbolic 3-manifold has a pair of non-hyperbolic Dehn fillings of a particular kind if and only if it is the double branched cover of one of a certain explicit family of tangles. We conclude with a sketch of the proof of one of these classification results, describing in particular how one shows that the fillings under consideration arise from tangle fillings.

I would like to thank Joan Licata for her help as Course Assistant for the lecture series, and Gabriel Drummond-Cole for producing his "real time" notes on the lectures. I would also like to thank Sungmo Kang for doing the figures and for his help with the proofreading.

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LECTURE 1

3-manifolds and knots

The main topic of these lectures is Dehn surgery. However, to provide a context for our discussion of this construction, we will first make some general remarks about 3-manifolds.

1.1. 3-manifolds

2- and 3-manifolds will always be compact, connected and orientable unless otherwise indicated. Also, we will always be working in the PL or smooth category; these are equivalent in dimensions ≤ 3.

Every 3-manifold contains lots of surfaces, for example those contained in a co-ordinate neighborhood. So to get global information about a 3-manifold from surfaces contained in it, we need some restriction on the surfaces. One of the most useful of these is the requirement that the surfaces be incompressible. Let M be a 3-manifold, and let $F \subset M$ be a surface that is either *properly embedded*, i.e. $F \cap \partial M = \partial F$, or contained in ∂M. F is *compressible* if there is a disk $D \subset M$ such that $D \cap F = \partial D$ and ∂D is *essential* in F, i.e. does not bound a disk in F. Otherwise, F is *incompressible*.

This important concept was introduced by Haken [Ha]. It formed the basis of his theory of what are now called *Haken manifolds*, that is, irreducible 3-manifolds that contain incompressible surfaces.

One of the cornerstones of 3-dimensional topology is the following theorem of Papakyriakopoulos, which says that the geometric condition of incompressibility is equivalent to a purely homotopy-theoretic condition. (Historically, this theorem was a combination of two theorems, the Loop Theorem and Dehn's Lemma, the latter being famously announced by Dehn in 1910 [Deh] but with an erroneous proof. We call it the Disk Theorem because it says that the existence of a (suitably non-trivial) singular disk is equivalent to the existence of an (analogously non-trivial) embedded disk. In the same spirit, there are also the Sphere, Torus and Annulus Theorems.)

Disk Theorem (Papakyriakopoulos [P]). *$F \subset M$ is incompressible if and only if $\pi_1(F) \to \pi_1(M)$ is injective.*

Clearly there are some trivial incompressible surfaces that are not very useful. Eliminating these leads to the following definition. A properly embedded surface $F \subset M$ is *essential* if either

(1) $F \cong S^2$ and F does not bound a B^3 in M;
(2) $F \cong D^2$ and ∂F is essential in ∂M;
(3) $F \not\cong S^2$ or D^2 and is incompressible and not boundary parallel. ($F \subset M$ is *boundary parallel* if there is an embedding $F' \times I \subset M$ such that $F' \times \{0\} \subset \partial M$ and $F = F' \times \{1\} \cup \partial F' \times I$.)

A 3-manifold M is *irreducible* if it does not contain an essential S^2. The following result is fundamental.

3-Dimensional Schönflies Theorem (Alexander [**A1**]). *S^3 is irreducible.*

The 2-dimensional analog is the classical Schönflies Theorem. Alexander first announced a proof of the 3-dimensional version in complete generality (see [**A2**]) but soon gave counterexamples, including the famous Alexander horned sphere, [**A2**], [**A3**], while also proving [**A1**] that the result is true in the smooth category.

The surfaces of non-negative Euler characteristic are the sphere, disk, annulus and torus (S^2, D^2, A^2, T^2). These play a key role in the theory of 3-manifolds; in particular, every 3-manifold can be decomposed into canonical pieces by cutting it up along such surfaces.

First, spheres. If S is a disjoint union of 2-spheres in $\operatorname{int} M$, by *decomposing M along S* we mean the operation of cutting M along S and then capping off the resulting 2-sphere boundary components with 3-balls.

Kneser's Theorem (Kneser [**Kn**]). *Every oriented 3-manifold can be decomposed along a finite disjoint union of 2-spheres to give a collection of irreducible 3-manifolds $M_1, M_2, \ldots, M_n$. The M_i's are unique up to orientation-preserving homeomorphism and insertion or deletion of copies of S^3.*

The existence part of this theorem is a finiteness statement: you can't go on decomposing a 3-manifold along essential spheres forever. To prove it Kneser used a triangulation of the 3-manifold and introduced the idea of a *normal surface*; this idea later played an important role in Haken's work.

Kneser's theorem can be expressed in terms of the connected sum operation. If M_1 and M_2 are oriented 3-manifolds, their *connected sum* $M_1 \# M_2$ is defined to be the manifold obtained by removing the interior of a 3-ball $B_i \subset \operatorname{int} M_i$, $i = 1, 2$, and then gluing the resulting punctured manifolds together by an orientation-reversing homeomorphism $\partial B_1 \to \partial B_2$.

Note that $M \# S^3 \cong M$. If $M \cong M_1 \# M_2$ implies that M_1 or $M_2 \cong S^3$ then M is *prime*. An irreducible 3-manifold is prime, and a prime 3-manifold is either irreducible or $S^1 \times S^2$.

Prime Factorization Theorem (Kneser [**Kn**], Milnor [**M**]). *Every oriented 3-manifold is a connected sum of prime manifolds $M_1 \# M_2 \ldots \# M_n$, $M_i \not\cong S^3$. The summands M_i are unique up to order and orientation-preserving homeomorphism.*

For disks we have the following; see [**Bo**]. M is *∂-irreducible* if it does not contain an essential D^2 (i.e. ∂M is incompressible). Let M be an irreducible 3-manifold. Then M contains a 3-submanifold W, unique up to isotopy, such that $\partial M \subset W$, $\overline{M - W}$ is irreducible and ∂-irreducible, and $\overline{M - W}$ is obtained by cutting M along a maximal disjoint union of non-parallel essential disks in M (and then discarding any B^3 components).

For example, if M is a handlebody, then $W = M$. (In general W is a disjoint union of what are called *compression bodies*, one for each component of ∂M.)

A *Seifert fiber space* (SFS) is a 3-manifold M that is a disjoint union of circles (*fibers*), such that each fiber has a *fibered solid torus* neighborhood, i.e. $D^2 \times I$ with $D^2 \times \{0\}$ and $D^2 \times \{1\}$ identified by a rotation ρ through $\frac{2\pi p}{q}$ ($q \geq 1$, $(p, q) = 1$). The *fibers* are the images of the arc $(0, 0) \times I$ (the central fiber) and the union of the arcs $x \times I$, $\rho(x) \times I, \ldots, \rho^{q-1}(x) \times I$, $x \neq (0, 0)$. We will say that the fibers other than the central fiber are *(p,q)-curves* in the solid torus. If $q \geq 2$ the central fiber is an *exceptional fiber of multiplicity* q. The quotient space of M obtained by identifying each fiber to a point is a surface, the *base* surface of M.

Finally, for annuli and tori we have the following. A 3-manifold M is *simple* if it contains no essential surface of non-negative Euler characteristic.

JSJ-Decomposition Theorem (Jaco-Shalen [JS], Johannson [Jo]). *In an irreducible, ∂-irreducible 3-manifold M, there is a collection $\mathcal{F}$ of disjoint essential annuli and tori such that each component of M cut along $\mathcal{F}$ is either a SFS, an I-bundle over a surface, or simple. A minimal such collection is unique up to isotopy.*

If a SFS M has base surface F and n exceptional fibers, of multiplicities $q_1, \dots, q_n$, we shall say M is *of type* $F(q_1, \dots, q_n)$. A SFS is *small* if one of the following holds:

- $F \cong S^2, \quad n \leq 3$
- $F \cong D^2, \quad n \leq 2$
- $F \cong A^2, \quad n \leq 1$
- $F \cong P^2, \quad n \leq 1$
- $F \cong$ Möbius band, $\quad n = 0$
- $F \cong$ pair of pants, $\quad n = 0$.

The SFS's of type $S^2(q_1, q_2)$ are S^3, the lens spaces, and $S^1 \times S^2$. Those of type $S^2(q_1, q_2, q_3)$ with $\sum \frac{1}{q_i} > 1$ (i.e. where $\{q_1, q_2, q_3\}$ is one of the Platonic triples $\{2,2,n\}$, $\{2,3,3\}$, $\{2,3,4\}$, or $\{2,3,5\}$) are the round (elliptic) 3-manifolds other than S^3 and the lens spaces. By the Geometrization Conjecture (see below), these manifolds, together with S^3 and the lens spaces, are precisely the closed 3-manifolds with finite fundamental group.

If a SFS is not small then it contains an essential T^2.

Lemma 1.1. *Let M be an irreducible 3-manifold whose boundary consists of tori. If M contains an essential A^2 then M either contains an essential T^2 or is a small SFS.*

For simple 3-manifolds we have the following, which is essentially the Geometrization Conjecture.

Theorem 1.2 (Thurston [Th], if $\partial M \neq \emptyset$; Perelman [**Pe1**], [**Pe2**], [**Pe3**], if $\partial M = \emptyset$). *M is simple if and only if either*

(1) *$M_0 = M -$(torus components of ∂M) has a complete hyperbolic structure with ∂M_0 totally geodesic; or*
(2) *M is a closed small SFS; or*
(3) *$M \cong B^3$.*

1.2. Knots

Let K be a knot in S^3. The *exterior* of K is $M_K = S^3 - \operatorname{int} N(K)$, where $N(K)$ is a regular neighborhood of K. Let us consider the manifold M_K in light of the discussion in the previous section.

First, by the 3-Dimensional Schönflies Theorem, M_K is always irreducible.

Second, M_K is ∂-reducible $\iff M_K \cong S^1 \times D^2 \iff K = U$, the unknot.

A *torus knot* is a non-trivial knot K that lies on a Heegaard torus T in S^3. If K is a (p,q)-curve with respect to the natural co-ordinate system on T, we write $K = T_{p,q}$ ($|p|, |q| > 1$). The exterior of $T_{p,q}$ is a SFS of type $D^2(|p|, |q|)$.

Let J be a knot in $\text{int}(S^1 \times D^2)$, that is not isotopic to $S^1 \times *$ and does not lie in a 3-ball. Let K_0 be a non-trivial knot in S^3, and let $h : S^1 \times D^2 \to N(K_0)$ be a homeomorphism. Then $K = h(J)$ is a *satellite* of K_0. See Figure 1.1. Note that the torus $\partial N(K_0)$ is essential in M_K.

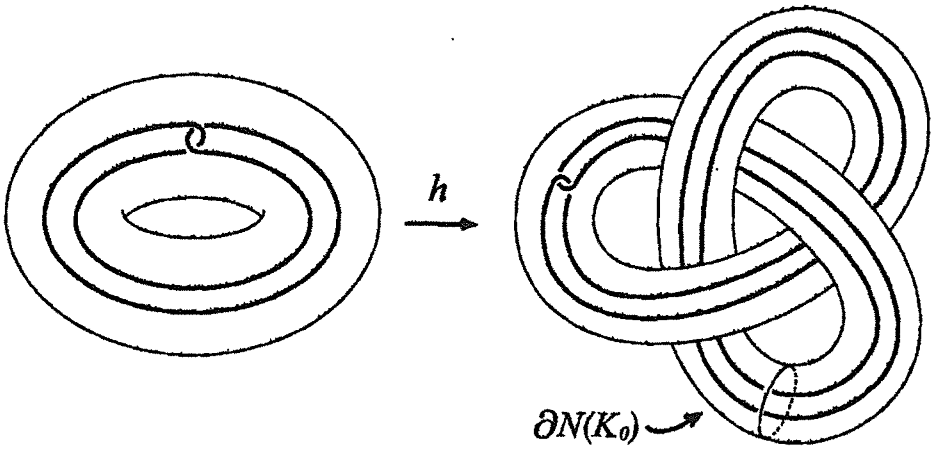


FIGURE 1.1

A special case is when $J = C_{p,q}$, a (p,q)-curve in $\text{int}(S^1 \times D^2)$, where $q \geq 2$. (See Figure 1.2, which shows the case $p = 3$, $q = 4$.) Then (choosing h to take $S^1 \times *$, $* \in \partial D^2$, to a longitude of K_0) $K = h(C_{p,q})$ is the (p,q)-*cable* of K_0.

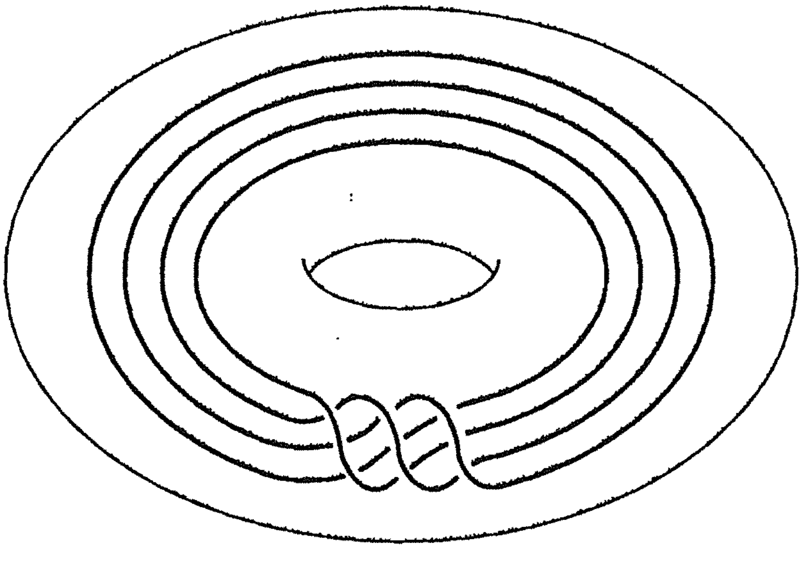


FIGURE 1.2

The following theorem is an easy consequence of the Disk Theorem and the 3-Dimensional Schönflies Theorem. However, Alexander's proof did not use the former.

Theorem 1.3 (Alexander [A1]). *Every $T^2 \subset S^3$ bounds a solid torus.*

An immediate consequence is

Corollary 1.4. *M_K contains an essential T^2 if and only if K is a satellite knot.*

By Lemma 1.1, if M_K contains an essential annulus then either M_K contains an essential torus or M_K is a SFS over D^2 with two exceptional fibers, i.e. K is a torus knot.

M_K is simple if and only if $\operatorname{int} M_K$ has a complete hyperbolic structure, in which case we say K is *hyperbolic.*

Summarizing, we have

Theorem 1.5. *Let K be a knot in S^3. Then exactly one of the following holds.*

(1) *K is the unknot* *(M_K contains an essential D^2);*
(2) *K is a torus knot* *(M_K contains an essential A^2 but no essential T^2);*
(3) *K is a satellite knot* *(M_K contains an essential T^2);*
(4) *K is hyperbolic* *(M_K is simple).*

1.3. Exercises

1. Let γ be a simple closed curve in a surface F. Show that γ is null-homotopic if and only if γ bounds a disk in F.
2. Show that every $T^2 \subset S^3$ bounds a solid torus,
   (a) using the Disk Theorem;
   (b) without using the Disk Theorem.
3. For any $g \geq 2$, find an example of a closed surface of genus g in S^3 that does not bound a handlebody.
4. Let M be an irreducible 3-manifold and T a torus component of ∂M. Show that if T is compressible then M is a solid torus.
5. Let K be a knot. Show that the following are equivalent:
   (a) K is the unknot;
   (b) $M_K \cong S^1 \times D^2$;
   (c) M_K is ∂-reducible;
   (d) $\pi_1(M_K) \cong \mathbb{Z}$.
6. Extending the definition of compressibility to not necessarily connected surfaces, show that $F \subset M$ is compressible if and only if some component of F is compressible.
7. Let M_1, M_2 be 3-manifolds, with $F_i \subset \partial M_i$, $i = 1, 2$. Let $h : F_1 \to F_2$ be a homeomorphism and define $M = M_1 \cup_h M_2$. Show that
   (a) if M_i is irreducible and F_i is incompressible, $i = 1, 2$, then M is irreducible.
   (b) if M_i is ∂-irreducible, F_i is incompressible, and no component of F_i is a disk, $i = 1, 2$, then M is ∂-irreducible.
8. Let T be a compressible torus in an irreducible 3-manifold M. Show that T either bounds a solid torus or lies in a 3-ball.
9. Let K be a satellite of a (non-trivial) knot K_0. Show that the torus ∂M_{K_0} is essential in M_K.
10. Show that the only 3-manifold that is prime but not irreducible is $S^1 \times S^2$.
11. Let M be an irreducible 3-manifold whose boundary consists of tori. Show that if M contains an essential A^2 then M either contains an essential T^2 or is a small SFS.
12. Let K_1 and K_2 be non-trivial knots. Show that their connected sum $K_1 \# K_2$ is a satellite of K_i, $i = 1, 2$.

Dehn surgery

2.1. Overview

Let K be a knot in S^3, with regular neighborhood $N(K) \cong K \times D^2$; $\mu = * \times \partial D^2$ is a *meridian* of K. Then $H_1(M_K) \cong \mathbb{Z}$, generated by $[\mu]$. Let λ be a *longitude* of K, i.e. a simple closed curve on $\partial N(K)$ having intersection number ± 1 with μ, and such that $[\lambda] = 0 \in H_1(M_K)$. Orient μ and λ so that $\mu \cdot \lambda = -\lambda \cdot \mu = 1$ on $\partial N(K)$. Then $[\mu]$ and $[\lambda]$ are a basis for $H_1(\partial M_K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

If α, β are unoriented simple closed curves on T^2 then α and β are isotopic if and only if $[\alpha] = \pm[\beta] \in H_1(T^2)$. A *slope* on T^2 is an isotopy class of unoriented essential simple closed curves on T^2. The *distance* $\Delta(\alpha, \beta)$ between two slopes is their minimal geometric intersection number.

If α is a slope on ∂M_K, then $[\alpha] = \pm(m[\mu] + \ell[\lambda]) \in H_1(\partial M_K)$ for some coprime integers m and ℓ. The correspondence $\alpha \leftrightarrow m/\ell$ sets up a bijection $\{\text{slopes on } \partial M_K\} \leftrightarrow \mathbb{Q} \cup \{1/0\}$. Note that $\Delta(m/\ell, m'/\ell') = |m\ell' - m'\ell|$.

$K(\alpha) = K(m/\ell) = \alpha$-(or m/ℓ-) *Dehn surgery on* $K = M_K \cup S^1 \times D^2$, glued along their boundaries in such a way that α is identified with $* \times \partial D^2$, a meridian of $S^1 \times D^2$.

We have $K(1/0) \cong S^3$ for all K (the *trivial* Dehn surgery). Also, $H_1(K(m/\ell)) \cong \mathbb{Z}_m$.

Dehn surgery was introduced by Dehn in 1910 [**Deh**]. The historical background to this was Poincaré's example, the spherical dodecahedral space D, of a non-simply-connected homology 3-sphere [**Po**]. Dehn pointed out that one can also obtain such manifolds by taking $m = 1$ (and $\ell \neq 0$) in the above construction. In fact, it turns out that D can be obtained by 1-surgery on the right-handed trefoil.

Corresponding to the four cases in Theorem 1.5 we have the following. Part (2) is due to Moser [**Mos**], part (3) is due to Berge [**Be1**] and Gabai [**Ga1**] (we will say more about this in Lecture 3), and part (4) is due to Thurston [**Th**].

Theorem 2.1.

(1) $U(m/\ell) = -L(m, \ell)$.

(2) *Let* $d = \Delta(m/\ell, pq/1) = |m - \ell pq|$. *Then*

$$T_{p,q}(m/\ell) = \begin{cases} S^2(p,q,d)\ , & \text{if } d > 1 \\ L(m, \ell q^2)\ , & \text{if } d = 1 \\ L(p,q) \# L(q,p)\ , & \text{if } d = 0 \end{cases}$$

(3) *Let K be a satellite knot, constructed from $J \subset \operatorname{int}(S^1 \times D^2)$, and let T be the corresponding essential torus in M_K. Then T usually remains incompressible in $K(\alpha)$. More precisely:*

(i) *If T compresses in $K(\alpha)$ for infinitely many α then J is a (p,q)-curve.*

(ii) *The curves J that are not (p,q)-curves such that T compresses in $K(\alpha)$ for some $\alpha \neq \mu$ are completely classified. Moreover, α is integral, and is unique except for exactly one curve J (up to homeomorphism of $S^1 \times D^2$), for which there are two such α, these being consecutive integers.*

(4) *Let K be hyperbolic. Then $K(\alpha)$ is hyperbolic for all but finitely many α.*

We make some comments on parts (2), (3) and (4) of Theorem 2.1.

(2) Let K be the torus knot $T_{p,q}$. Then K lies on a Heegaard torus $T \subset S^3$, which separates S^3 into two solid tori X and X': $S^3 = X \cup_T X'$. Note that K is a (p,q)-curve on X and a (q,p)-curve on X'. Let $N(K)$ be a regular neighborhood of K in S^3 such that $N(K) \cap T$ is a regular neighborhood of K in T. Thus $A = \overline{T - N(K)}$ is an annulus, and $M_K = S^3 - \operatorname{int} N(K) \cong X \cup_A X'$; see Figure 2.1.

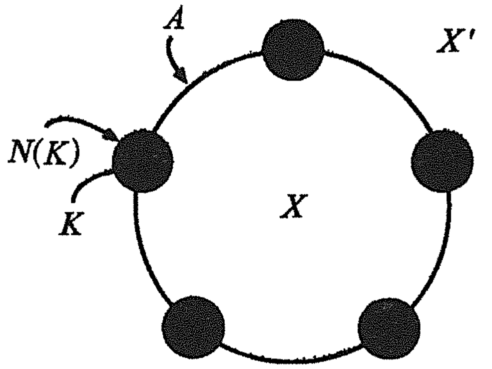


FIGURE 2.1

Now X (resp. X') has a Seifert fibering such that the ordinary fibers are (p,q)-curves (resp. (q,p)-curves), and we may choose these Seifert fiberings to agree on the annulus A. We thus see that M_K is a SFS of type $D^2(|p|,|q|)$.

Let us now compute the framing on K induced from T. By definition this is the linking number $\ell k(K',K)$, where K' is a parallel copy of K on T. Let X^+ be a larger concentric copy of X, with meridian and longitude μ^+ and λ^+. Then K' is isotopic in $S^3 - K$ to K^+, a (p,q)-curve on ∂X^+. Note that K^+ is homologous on ∂X^+ to $p\mu^+ + q\lambda^+$. Also, $\ell k(\lambda^+, K) = 0$, since $\lambda^+ \sim 0$ in $S^3 - K$, while $\ell k(\mu^+, K) = q$. Therefore $\ell k(K',K) = \ell k(K^+,K) = pq$.

Thus on ∂M_K, the Seifert fibers have slope pq. Now consider m/ℓ-Dehn surgery on K, $K(m/\ell) = M_K \cup V$, V a solid torus. If the fibers on ∂M_K are not meridians of V, then the fibering of ∂M_K can be extended to a Seifert fibering of V, where the multiplicity of the central fiber is $d = \Delta(m/\ell, pq/1)$. This gives the first two parts of (2).

We discuss the third part of (2), i.e. the case $T_{p,q}(pq)$, in Section 2.2, as a special case of a more general situation.

(3) Let us first look at the case of cable knots. Here we have a solid torus V and J is a (p,q)-curve $C_{p,q} \subset \operatorname{int} V$. Let N be a regular neighborhood of $C_{p,q}$ in V, and let $Y = \overline{V - N}$. There is an annulus $A \subset Y$ with one boundary component a (p,q)-curve on ∂V, and the other a $(pq,1)$-curve on ∂N. See Figure 2.2. Let μ, λ

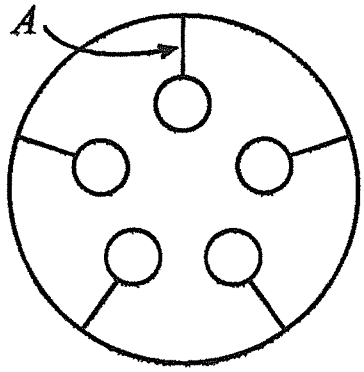


FIGURE 2.2

be a meridian-longitude pair on ∂N. Let $t : Y \to Y$ be a Dehn twist along A. Then $t^\ell(\mu) = \mu + \ell(pq\mu + \lambda) = (\ell pq+1)\mu + \ell\lambda$, a curve of slope $(\ell pq+1)/\ell$ on ∂N. Hence $(\ell pq + 1)/\ell$-Dehn surgery on $C_{p,q}$ in V gives a solid torus again. Thus $T = \partial V$ compresses in $K(\alpha)$ for infinitely many slopes α. Conversely, if this happens then J is a (p, q)-curve by [**CGLS**, Theorem 2.4.4].

We will discuss part (ii) in Section 3.5 of Lecture 3.

(4) There has been a lot of work done trying to understand the exceptions to this general principle. We shall say more about this in the subsequent lectures, but right now let's look at an example.

Example

Consider the simplest hyperbolic knot, the figure-8 knot K. Then M_K is a punctured T^2-bundle over S^1. So $K(0)$ is a T^2-bundle over S^1, and hence *toroidal* (i.e. contains an essential torus).

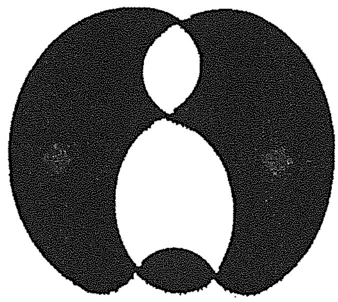


FIGURE 2.3

M_K contains a once-punctured Klein bottle with boundary slope 4; see Figure 2.3. Hence $K(4)$ contains a Klein bottle, and so is non-hyperbolic. (In fact, it is toroidal.) Since K is amphicheiral, $K(m/\ell)$ is homeomorphic (by an orientation-reversing homeomorphism) to $K(-m/\ell)$. So $K(-4) \cong K(4)$. We shall see in Lecture 4 that $K(\pm 1)$, $K(\pm 2)$ and $K(\pm 3)$ are also non-hyperbolic, more precisely that they are SFS's of type $S^2(q_1, q_2, q_3)$.

2.2. Framed surgery on knots on surfaces

We now describe a systematic way of constructing knots in S^3 with certain interesting integral surgeries.

If X is a 3-manifold and K is a simple closed curve on ∂X, let $X[K]$ denote the 3-manifold obtained by attaching a 2-handle to X along K. Then $\partial X[K]$ is the surface obtained by surgering ∂X along K.

Let F be a closed surface in S^3, separating S^3 into X and X', say. Let K be a knot that lies on F, and let $m \in \mathbb{Z}$ be the framing of K induced by F. Then $K(m) \cong X[K] \cup_\partial X'[K]$.

This can be seen as follows; see Figure 2.4. Let $F_0 = F - \operatorname{int} N(K)$, where $N(K)$ is chosen so that $N(K) \cap F$ is a regular neighborhood of K in F; so $\partial F_0 = \alpha_1 \cup \alpha_2$, two parallel curves on ∂M_K with slope m. Then $M_K \cong X \cup_{F_0} X'$. Now $K(m) = M_K \cup V$, V a solid torus. Note that α_i bounds a disk $D_i \subset V$, $i = 1, 2$. α_1 and α_2 cut ∂V into two annuli A, A', and D_1 and D_2 cut V into two 3-balls H, H', where $A = \partial V \cap X \subset \partial H$, and $A' = \partial V \cap X' \subset \partial H'$.

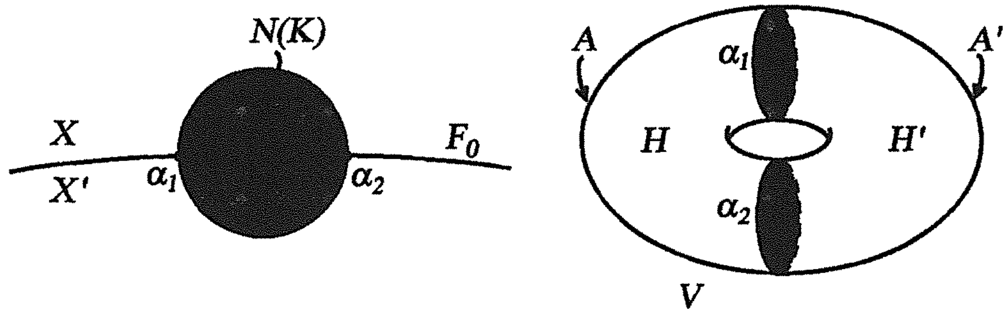


FIGURE 2.4

Then

$$\begin{aligned} K(m) &\cong (X \cup_{F_0} X') \cup_\partial (H \cup_{D_1 \cup D_2} H') \\ &= (X \cup_A H) \cup_\partial (X' \cup_{A'} H') \\ &= X[K] \cup_\partial X'[K] \,. \end{aligned}$$

Note that $\partial X[K] = \partial X'[K] = \widehat{F}_0 = F_0 \cup D_1 \cup D_2$, and the core K_m of V meets the surface $\widehat{F}_0$ in two points.

Examples

(1) The torus knot $K = T_{p,q}$ lies on a Heegaard torus F in S^3. The induced framing is pq. Here X and X' are solid tori, and K is a (p,q)-curve on (say) X and a (q,p)-curve on X'. Hence $X[K] \cong L(q,p) - \operatorname{int} B^3$, $X'[K] \cong L(p,q) - \operatorname{int} B^3$, and so $T_{p,q}(pq) \cong L(p,q) \# L(q,p)$.

(2) Let K be the (p,q)-cable of a knot K_0, $q \geq 2$. If $C_{p,q}$ is a (p,q)-curve in the interior of a solid torus V then $C_{p,q}(pq) \cong S^1 \times D^2 \# L(q,p)$, the meridian of $S^1 \times D^2$ having slope p/q with respect to the usual co-ordinates on ∂V. Hence

$$K(pq) \cong K_0(p/q) \# L(q,p) \,.$$

If $K_0 = U$ then $K = T_{p,q}$ and we get example (1). If $K_0 \neq U$ then $K_0(p/q) \not\cong S^3$, so $K(pq)$ is reducible.

This brings us to what is arguably the most important open problem on Dehn surgery, the Cabling Conjecture, formulated by González-Acuña and Short in 1986.

Cabling Conjecture (González-Acuña-Short [**GS**]). *If K is a non-trivial knot such that $K(\alpha)$ is reducible for some α then K is a cable knot.*

(It is convenient here to regard a torus knot as a cable of the unknot.)

The Cabling Conjecture is known to hold for several classes of knots, for example: alternating knots [**MT**], strongly invertible knots [**E1**], symmetric knots [**HS**], arborescent knots [**W3**], knots of bridge number at most 4 [**Ho**], and satellite knots [**Sc**]. It follows from the last result that it is equivalent to the assertion that if K is hyperbolic then $K(\alpha)$ is always irreducible.

Finally, it is known that if $K(\alpha)$ is reducible then α is integral [**GL1**], and $K(\alpha)$ has a lens space summand [**GL2**]. The latter implies in particular that $K(0)$ is always irreducible, a fact first proved by Gabai [**Ga2**].

(3) The example of 4-surgery on the figure-8 knot mentioned earlier can be interpreted as framed surgery on a knot on a surface. Let S be the once-punctured Klein bottle with $\partial S = K$ shown in Figure 2.3. Let X be a (twisted) I-bundle neighborhood of S; so X is a handlebody of genus 2. Since S comes from the black regions of a black/white shading of a planar diagram of K, it is clear that $X' = \overline{S^3 - X}$ is also a handlebody. Hence $F = \partial X$ is a genus 2 Heegaard surface in S^3, and K lies on F with induced framing 4. Now $X[K]$ is a twisted I-bundle over the Klein bottle, which can also be described as a SFS of type $D^2(2,2)$. By carefully drawing K as it lies on $\partial X'$, one can see that $X'[K]$ is homeomorphic to the exterior of the trefoil. Hence $K(4)$ is a toroidal graph-manifold, of the form $D^2(2,2) \cup_\partial D^2(2,3)$.

2.3. Exercises

1. Show that $\Delta(m/\ell,\, m'/\ell') = |m\ell' - m'\ell|$.
2. Show that (up to orientation-preserving homeomorphism), $M(\alpha)$ depends only on the slope α.
3. Show that a SFS of type $D^2(2,2)$ is an I-bundle over the Klein bottle. (Hint: Note that $S^1 \times D^2$ is an I-bundle over the Möbius band.)
4. Let M be an irreducible 3-manifold that contains a Klein bottle F. Show that M is either
   (a) toroidal;
   (b) a twisted I-bundle over F; or
   (c) a SFS of type $S^2(2,2)$ or $S^2(2,2,n)$.
5. Which lens spaces $L(m,\ell)$ are SFS's of type $S^2(2,2)$?
6. Show that the exterior of $C_{p,q}$ in $S^1 \times D^2$ is a SFS of type $A^2(q)$.
7. Show that the boundary slope of the surface shown in Figure 2.3 is 4.
8. Show that for every $m \in \mathbb{Z}$ there exists a hyperbolic knot K such that $K(4m)$ is toroidal. What about $2m$? m?
9. Recall that for any $\ell \in \mathbb{Z}$, $(\ell pq + 1)/\ell$-Dehn surgery on $C_{p,q}$ in the interior of a solid torus V gives a solid torus V'. What's the slope of the meridian of V' in terms of the meridian-longitude co-ordinates of V?
10. Verify the details of Example (2) in Section 2.2.

Exceptional Dehn surgeries

3.1. Exceptional surgeries

By an *exceptional Dehn surgery* we mean a pair $(K;\alpha)$ where K is a hyperbolic knot in S^3 and $\alpha \neq \mu$ is a slope such that $K(\alpha)$ is not hyperbolic. Such pairs tend to be quite rare, so it is perhaps not unreasonable to try to classify them. In this lecture we will describe what is known along these lines.

First we note that if $K(\alpha)$ is not hyperbolic, then by the Geometrization Conjecture it is either

(1) S^3
(2) $S^1 \times S^2$
(3) non-prime
(4) a lens space
(5) a SFS of type $S^2(q_1, q_2, q_3)$
(6) toroidal.

(1) and (2) never happen, by [**GL2**] and [**Ga2**], respectively; (3) is conjectured to never happen (this would follow from the Cabling Conjecture); (4), (5) and (6) all do occur.

Let us see how we can obtain knots K with (integral) surgeries $K(m)$ of types (4), (5) and (6) using the construction described in Lecture 2, Section 2.2. Although this construction only yields integral surgeries, it nevertheless accounts for many of the known exceptional surgeries on knots. That most exceptional surgeries turn out to be integral is a reflection of the fact that if $M(\alpha)$ and $M(\beta)$ are non-hyperbolic Dehn fillings on a hyperbolic 3-manifold M (see Lecture 4), then the distance $\Delta(\alpha,\beta)$ tends to be small. Since $K(\mu) \cong S^3$ is non-hyperbolic, where μ is the meridian of K, one would therefore expect that if $(K;\alpha) = (K; m/\ell)$ is exceptional then $\Delta(\alpha,\mu) = \ell$ is small. In fact, for all known examples, $\ell = 1$ or 2, and the only known examples with $\ell = 2$ are the knots with half-integral toroidal surgeries constructed by Eudave-Muñoz [**E2**] that we will say more about later.

3.2. Lens space surgeries

We first describe the knots with lens space surgeries constructed by Berge [**Be2**]. The following lemma is relevant.

Lemma 3.1. *Let X be a handlebody of genus $n \geq 1$, and let K be a simple closed curve in ∂X. The following are equivalent:*

(i) *$X[K]$ is a handlebody of genus $n-1$.*
(ii) *There is a disk $D \subset X$ such that K meets ∂D transversely in a single point.*
(iii) *$[K]$ belongs to a basis for the free group $\pi_1(X)$.*

If these conditions are satisfied we say K is *primitive*. Now in the construction described in Section 2.2 take F to be a genus 2 Heegaard surface in S^3, so X and X' are genus 2 handlebodies, and suppose $K \subset F$ is *doubly primitive*, i.e. K is primitive in X and X'. Then $K(m) \cong X[K] \cup_\partial X'[K]$ is a union of two solid tori, and hence a lens space. (Recall that if $K \neq U$ then $K(m)$ cannot be S^3 or $S^1 \times S^2$.)

The doubly primitive knots have been explicitly determined by Berge (see [**Be2**]), and are referred to as the *Berge knots*.

Berge Conjecture (Berge [**Be2**]). *A hyperbolic knot K has a lens space surgery if and only if K is a Berge knot, and the surgery is the corresponding integral surgery.*

The Cyclic Surgery Theorem [**CGLS**] implies that at least the surgery must be integral.

Theorem 3.2 (Culler-Gordon-Luecke-Shalen [**CGLS**]). *Any lens space surgery on a hyperbolic knot is integral.*

Ozsváth and Szabó [**OS**], using their Heegaard Floer homology theory, give a necessary condition on the Alexander polynomial of K for the lens space $L(p,q)$ to arise as some surgery on K. Using this, they have verified that for $p \leq 1500$, the lens spaces $L(p,q)$ that can be obtained by integral surgery on a knot are precisely the lens spaces that are listed by Berge [**Be2**] as arising from his construction.

Bleiler and Litherland [**BL**] conjectured that no hyperbolic knot has a lens space surgery $L(p,q)$ with $p < 18$; this would follow from the Berge Conjecture. Baker [**Ba3**] has shown that the Bleiler-Litherland conjecture is true, with the one possible exception of $L(14, 11)$.

3.3. Seifert fiber space surgeries

If X is a handlebody of genus 2, say $K \subset \partial X$ is *Seifert* if $X[K]$ is a SFS other than $S^1 \times D^2$.

Then a knot K is *primitive/Seifert* if it lies on a genus 2 Heegaard surface F in S^3 and is primitive in one of the complementary genus 2 handlebodies, say X, and Seifert in the other, X'. Then $K(m) \cong X[K] \cup_\partial X'[K]$ is obtained by gluing a solid torus to a SFS, and hence is a SFS. (A priori, $K(m)$ could also be a connected sum of lens spaces, but according to the Cabling Conjecture this never happens if K is hyperbolic.)

Primitive/Seifert knots are studied by Dean in [**D**], and are called *Dean knots*. Most known knots with SFS surgeries are Dean knots; however, there are some that are not.

If $X[K]$ is a SFS and not a solid torus, then one can show that it is of the form $D^2(q_1, q_2)$ or $M^2(q)$, where M^2 denotes the Möbius band. Hence if K is a Dean knot, then the corresponding surgery $K(m)$ (if not a lens space) is of the form $S^2(q_1, q_2, q_3)$ or $P^2(q_1, q_2)$. Note that those of type $P^2(q_1, q_2)$ are toroidal. Two infinite families of hyperbolic knots with surgeries of this latter type are given by Eudave-Muñoz in [**E3**]; these are Dean knots. We remark that even though there are hyperbolic knots with SFS surgeries that are not Dean knots, in all known examples the Seifert fiber spaces obtained are of one of the two above types.

Question. *Let M be a Seifert fiber space that arises as Dehn surgery on a hyperbolic knot, and suppose M is not a lens space. Is M of type $S^2(q_1, q_2, q_3)$ or $P^2(q_1, q_2)$?*

The first examples of hyperbolic knots with SFS surgeries that are not Dean knots were given by Mattman, Miyazaki and Motegi [**MMM**]. These knots still lie on a genus 2 Heegaard surface, but as separating curves. It is shown in [**MMM**] that they do not have tunnel number 1. Since it is easy to see that any knot that is primitive on a genus 2 Heegaard handlebody in S^3 has tunnel number 1, it follows that they are not Dean knots. More recently, examples of hyperbolic knots with SFS surgeries that do not even lie on a genus 2 Heegaard surface are given in [**DMM**].

The main open problem about SFS surgeries is the generalization of Theorem 3.2.

Conjecture 3.3. *Any Seifert fibered surgery on a hyperbolic knot is integral.*

For toroidal Seifert fibered surgeries, this has been proved by Boyer and Zhang [**BZ2**].

We remark that Motegi and Song have shown that every integer occurs as the slope of a SFS surgery on a hyperbolic knot [**MS**].

3.4. Toroidal surgeries

Again let F be a genus 2 Heegaard surface in S^3, and let K be a non-separating simple closed curve on F, with induced framing m. Then, as before, $K(m) \cong X[K] \cup_T X'[K]$, where $T = \partial X[K] = \partial X'[K]$ is a torus. In contrast to cases (4) and (5) above, here we want T to be incompressible in $X[K]$ and $X'[K]$, and hence in $K(m)$. A criterion for this is given by the following result of Jaco [**J**].

Handle Addition Lemma (Jaco [**J**]). *Let X be a ∂-reducible 3-manifold with connected boundary and K a simple closed curve on ∂X such that $\partial X - K$ is incompressible in X. Then $X[K]$ is ∂-irreducible.*

Hence if $F - K$ is incompressible in X and X' then $K(m)$ is toroidal. It is clear that lots of examples can be constructed in this way. (See [**Te1**] for instance, which shows that for any integer m there exists a hyperbolic knot K such that $K(m)$ is toroidal.)

On the other hand there are hyperbolic knots with integral toroidal surgeries that do not arise from this construction: it is not hard to see that any knot K that lies on a genus 2 Heegaard surface has tunnel number $t(K)$ at most 2, whereas Eudave-Muñoz and Luecke have shown that there are hyperbolic knots with integral toroidal surgeries with $t(K)$ arbitrarily large [**EL**].

As we mentioned in Section 3.1, there also exist non-integral toroidal surgeries on hyperbolic knots. However, these have been completely classified; see Theorem 5.3 in Lecture 5.

3.5. Knots in solid tori

We return to part (3) of Theorem 2.1, dealing with Dehn surgery on satellite knots. So let K be a satellite knot, constructed from a non-trivial knot K_0 and a curve $J \subset \text{int}(S^1 \times D^2)$ as described in Section 1.2. Then $T = \partial M_{K_0}$ is an essential torus in M_K. Now $K(\alpha) = M_{K_0} \cup_T J(\alpha)$, where $J(\alpha)$ is the result of α-Dehn surgery on $J \subset S^1 \times D^2$, and hence, since T is incompressible in M_{K_0}, it will be incompressible in $K(\alpha)$ unless it compresses in $J(\alpha)$. This leads to the question:

For which knots J in $S^1 \times D^2$ is $J(\alpha)$ ∂-reducible for some $\alpha \neq \mu$?

(We assume as usual that J is not a core of $S^1 \times D^2$ and does not lie in a 3-ball.)

Three remarks: first, Scharlemann has shown [**Sc**] that, if J is not a (p,q)-curve $C_{p,q}$, then $J(\alpha)$ is irreducible, and so $J(\alpha)$ will be ∂-reducible if and only if it is a solid torus. Second, we have seen in Section 2.1 that if J is a (p,q)-curve then $J(\alpha)$ is a solid torus for infinitely many slopes α. Third, it is shown in [**CGLS**, Theorem 2.4.4] that if J is not a (p,q)-curve, and $J(\alpha)$ and $J(\beta)$ are ∂-reducible, then $\Delta(\alpha,\beta) \leq 1$. In particular, $\Delta(\alpha,\mu) \leq 1$.

A method for constructing knots J in $S^1 \times D^2$ with non-trivial solid torus surgeries has been given by Berge [**Be1**]. Here is his construction. Let X be a genus 2 handlebody, and let α, β be simple closed curves on ∂X, intersecting transversely in a single point, such that each is primitive in X. Let N be a regular neighborhood of $\alpha \cup \beta$ in ∂X (so N is a once-punctured torus), and let $\gamma = \partial N$. Let $Y = X[\gamma]$. Then ∂Y consists of two tori, one of which, say T_0, contains α and β. Then the Dehn filling $Y(\alpha) \cong X[\alpha] \cong S^1 \times D^2$, since α is primitive in X. Similarly $Y(\beta) \cong S^1 \times D^2$. Berge also completely classified the knots $J \subset S^1 \times D^2$ that arise from this construction [**Be1**]. (They fall into six types, I–VI. Type I consists of the (p,q)-curves, those of type II are certain cables of (p,q)-curves, while the exteriors of those of types III–VI are simple.) Denote the set of such knots by $\mathbb{J}$.

Now Gabai showed in [**Ga1**] that if $J \subset S^1 \times D^2$ has a non-trivial solid torus surgery $J(\alpha)$ then J is a 1-bridge braid. Furthermore, one can see that a 1-bridge braid with a non-trivial solid torus surgery must arise from Berge's construction. So putting all this together one gets a complete description of all knots in $S^1 \times D^2$ with non-trivial solid torus surgeries.

Theorem 3.4 (Berge [**Be1**], Gabai [**Ga1**]). *A knot J in $S^1 \times D^2$ has a non-trivial Dehn surgery yielding $S^1 \times D^2$ if and only if $J \in \mathbb{J}$.*

One interesting consequence of Berge's classification is that there is a unique knot J in $S^1 \times D^2$, that is not a (p,q)-curve, having more than one, and hence

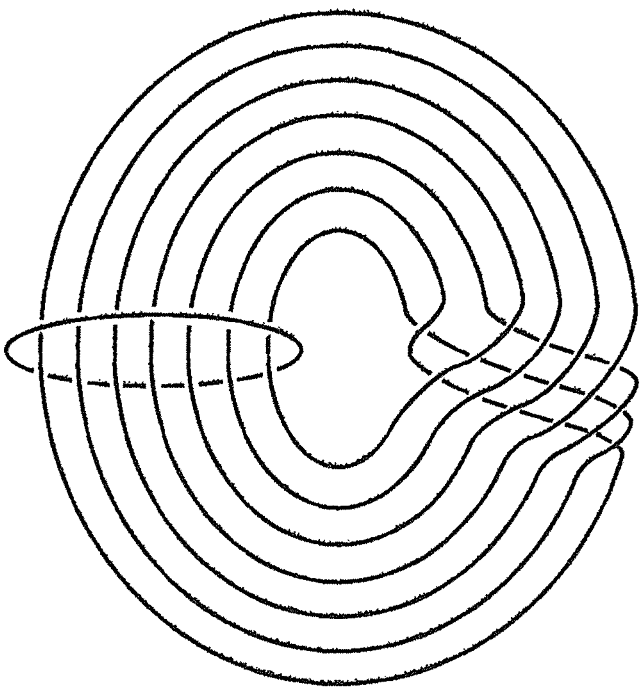


FIGURE 3.1

exactly two, non-trivial solid torus surgeries. The exterior of this knot in $S^1 \times D^2$ is the exterior of the 2-component link shown in Figure 3.1. (This manifold is called the *Berge manifold* in [**MP**].)

There is a connection here with the Berge knots discussed in Section 3.2. Suppose $J \in \mathbb{J}$, so that $J(\alpha)$ is a solid torus for some $\alpha \neq \mu$. Then, for any unknotted embedding $h : S^1 \times D^2 \to S^3$, $h(J)$ is a knot K in S^3 and the surgery on K corresponding to α is a lens space. Since h may be precomposed with any power of a Dehn twist along the meridian disk of $S^1 \times D^2$, J gives rise to infinitely many knots in S^3 with lens space surgeries. In this way the knots in $\mathbb{J}$ of types I–VI give rise to Berge knots of types I–VI. (The Berge knots of types I and II are the torus knots and certain cables of torus knots, respectively.) However, these do not account for all knots with lens space surgeries. Berge describes six additional families of such knots, of types VII–XII. Those of types VII and VIII lie on a fiber of the trefoil or figure-8 knot, respectively, while the remaining types IX–XII Berge calls "sporadic".

3.6. Exercises

1. Let X be a handlebody of genus n, and let K be a simple closed curve on ∂X such that there is a disk in X whose boundary meets K transversely in a single point. Show that $X[K]$ is a handlebody of genus $n - 1$.
2. Show that the simple closed curve on the boundary of a genus 2 handlebody illustrated in Figure 3.2 is primitive.

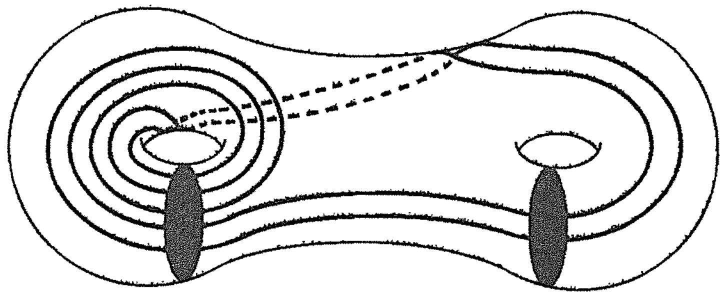


FIGURE 3.2

3. Show that the simple closed curve on the boundary of a genus 2 handlebody illustrated in Figure 3.3 is Seifert.

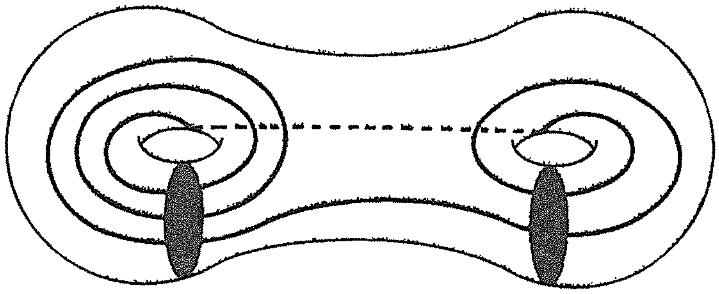


FIGURE 3.3

4. Show that a knot that lies on a genus g Heegaard surface in S^3 has tunnel number at most g.

. Let K be a simple closed curve on the boundary of a genus 2 handlebody X. Show that if $X[K]$ is a Seifert fiber space, then it is either a solid torus, or of the form $D^2(q_1, q_2)$ or $M^2(q)$, where M^2 is a Möbius band.

Rational tangle filling

4.1. Dehn filling

It is natural to extend the notion of Dehn surgery on a knot to that of Dehn filling on a 3-manifold along some torus boundary component. So let M be a 3-manifold with a torus boundary component T_0, and suppose that α is a slope on T_0. Define the *α-Dehn filling* on M, $M(\alpha)$, to be the manifold obtained by gluing a solid torus V to M so that the boundary of a meridional disc of V is glued to α:

$$M(\alpha) = M \cup_{T_0 = \partial V} V .$$

If M is hyperbolic in the sense of Theorem 1.2(1) (equivalently M is simple, i.e. contains no essential S^2, D^2, A^2, or T^2), then $M(\alpha)$ will usually be hyperbolic. We call the fillings for which this is not true *exceptional*.[1] The following theorem shows that it is unreasonable to try to describe all exceptional Dehn fillings.

Theorem 4.1 (Myers [My]). *Any 3-manifold N is of the form $M(\alpha)$ for some simple 3-manifold M and some slope α.*

However, experience shows that it is rare for a simple 3-manifold to have more than one exceptional filling, so if we define an *exceptional pair* $(M; \alpha, \beta)$ to be a simple 3-manifold M with non-hyperbolic fillings $M(\alpha)$ and $M(\beta)$, $\alpha \neq \beta$, then maybe we can classify all exceptional pairs. For example, when $M(\beta)$, say, is S^3, then we have the case of hyperbolic knots in S^3 discussed earlier.

In Lectures 2 and 3 we described one way to see certain integral Dehn surgeries on knots, by having the knot lie on a surface. Another way of constructing interesting Dehn surgeries, and more generally Dehn fillings, is by means of rational tangle surgery. In this construction, which is due to Montesinos [Mon], the relation with Dehn surgery comes about by passing to double branched covers. The advantage of rational tangle surgery is that tangles, and knots or links, are easy to visualize and properties of their double branched covers can often be explicitly read off from a diagram. The apparent drawback is that the manifolds you get (i.e. the double branched covers) always have a $\mathbb{Z}_2$-symmetry; in particular the knots in S^3 you get are always strongly invertible. (A link L is *strongly invertible* if there is an involution τ of S^3, with fixed-point set $\mathrm{Fix}(\tau)$ an unknotted circle, such that $\tau(L) = L$ and each component of L meets $\mathrm{Fix}(\tau)$ in two points.) Nevertheless, it turns out that (as we shall see in Lecture 6) many exceptional pairs arise in this way.

4.2. Tangles

A *tangle* is a pair (B, A), where B is S^3 minus the interiors of a finite number of disjoint 3-balls, and A is a properly embedded 1-manifold in B which meets

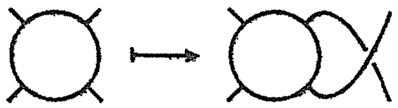
[1] With this terminology, $M_K(1/0) \cong S^3$ is an exceptional Dehn filling for any hyperbolic knot K.

each component of ∂B in four points. Note that our definition of tangle includes the case of a knot or link in S^3. A *marked* tangle is a tangle (B, A) with an identification of each pair $(S, S \cap A)$, where S is a component of ∂B, with $(S^2, Q = \{NE, NW, SW, SE\})$. Marked tangles (B, A) and (B, A') are *equal* if there is an isotopy of B, fixed on ∂B, taking A to A'.

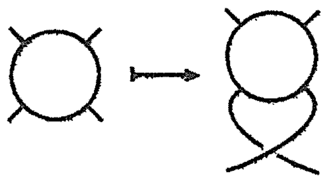
The *trivial* tangle is the tangle in B^3 homeomorphic to $(D^2, \textit{2 points}) \times I$.

Consider the following three operations on marked tangles in B^3:

$h =$ horizontal $\frac{1}{2}$-twist



$v =$ vertical $\frac{1}{2}$-twist



$r =$ reflection in the (NW/SE)-plane

Note that $rhr = v$.

Let $a_1, a_2, \ldots, a_k$ be a sequence of integers, $a_i \neq 0$, $2 \leq i \leq k$. Define the *rational tangle* $\mathcal{R}(a_1, a_2, \ldots, a_k)$ to be $(h^{a_1}r)(h^{a_2}r)\cdots(h^{a_k}r)\mathcal{R}(1/0)$, where $\mathcal{R}(1/0)$ is the tangle . Note that a rational tangle is trivial (as an unmarked tangle). Conversely, it can be shown that any such marked tangle is rational.

Let $p/q = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \frac{1}{a_k}}}$, where $(p, q) = 1$.

Theorem 4.2 (Conway [C]). *$\mathcal{R}(a_1, a_2, \ldots, a_k) = \mathcal{R}(a_1', a_2', \ldots, a_{k'}')$ if and only if $p/q = p'/q'$.*

We write $\mathcal{R}(a_1, a_2, \ldots, a_k) = \mathcal{R}(p/q)$.

Note that, since $rhr = v$,

$$\mathcal{R}(p/q) = \begin{cases} h^{a_1}v^{a_2}\cdots h^{a_{k-1}}v^{a_k}\mathcal{R}(1/0)\,, & k \text{ even} \\ h^{a_1}v^{a_2}\cdots v^{a_{k-1}}h^{a_k}\mathcal{R}(0/1)\,, & k \text{ odd} \end{cases}$$

where $\mathcal{R}(0/1) =$ .

Some examples of rational tangles are shown in Figure 4.1, corresponding to the rational numbers $1/4$, $-3/2$, $2/3$, $5/14$ and $2/3$ respectively.

Let $\mathcal{T} = (B, A)$ be a tangle. Since $H_1(B - A)$ is the free abelian group generated by the meridians of the components of A, there is a unique homomorphism $\pi_1(B - A) \to \mathbb{Z}_2$ sending each meridian to the non-trivial element of $\mathbb{Z}_2$. The corresponding double covering of $B - A$ can be completed to a branched covering of B with branch

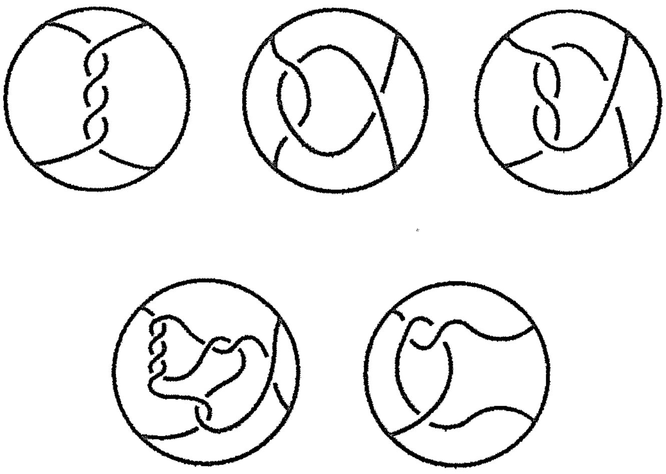


FIGURE 4.1

set A. We will denote this double branched covering by $\widetilde{\mathcal{T}}$. In particular, if $\mathcal{R}$ is a rational tangle then $\widetilde{\mathcal{R}}$ is a solid torus. See Figure 4.2.

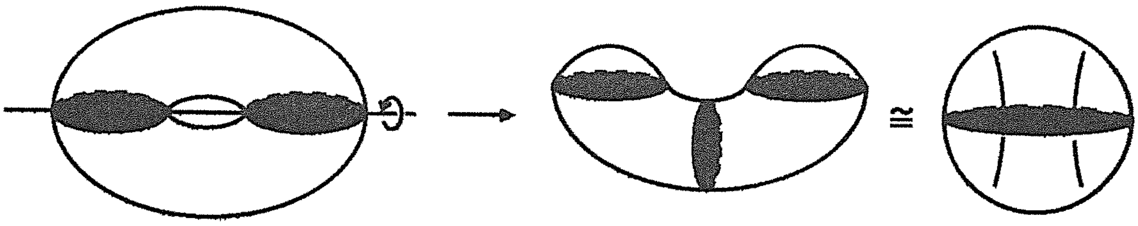
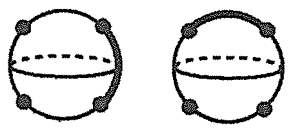


FIGURE 4.2

A *slope* on (S^2, Q) is the isotopy class (rel ∂) of an embedded arc τ in S^2 with $\partial\tau \subset Q$. Let $p : T^2 \to S^2$ be the double covering branched over Q. Then $\tilde{\tau} = p^{-1}(\tau)$ is an essential simple closed curve in T^2. Let μ, λ be the slopes on (S^2, Q):



μ λ

Orient $\tilde{\mu}$ and $\tilde{\lambda}$ so that on T^2 with its usual positive orientation $\tilde{\mu} \cdot \tilde{\lambda} = -1$. Then $[\tilde{\mu}], [\tilde{\lambda}]$ is a basis for $H_1(T^2)$ and we have bijections $\{\text{slopes on } (S^2, Q)\} \leftrightarrow \{\text{slopes on } T^2\} \leftrightarrow \mathbb{Q} \cup \{1/0\}$.

Theorem 4.3. *Under the above correspondence, the slope on the boundary of $\mathcal{R}(p/q)$ that lifts to a meridian of the solid torus $\widetilde{R(p/q)}$ is p/q.*

PROOF. Note that this is true for $\mathcal{R}(1/0)$.

Let $\tilde{h}, \tilde{r} : T^2 \to T^2$ be lifts of $h|S^2$, $r|S^2$. Then, with respect to the basis $[\tilde{\mu}], [\tilde{\lambda}]$, $\tilde{h}_*$ and $\tilde{r}_* : H_1(T^2) \to H_1(T^2)$ are given by the matrices $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ respectively. The meridian of the solid torus $\widetilde{R(p/q)}$ is $(\tilde{h}^{a_1}\tilde{r}) \dots (\tilde{h}^{a_k}\tilde{r})(\tilde{\mu})$, which (see Exercise 4.5 (2)) has co-ordinates

$$\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_k & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} e_{1,k} \\ e_{2,k} \end{bmatrix} = \begin{bmatrix} p \\ q \end{bmatrix} .$$

□

Corollary 4.4. *If $\mathcal{T}$ is a marked tangle then* $\widetilde{\mathcal{T}(p/q)} \cong \widetilde{\mathcal{T}}(-p/q)$.

Remark. The minus sign in Corollary 4.4 is because of the usual convention for parametrizing Dehn surgery slopes. One could argue that this should be reversed, thereby getting rid of the minus signs in both Corollary 4.4 and Theorem 2.1(1). Alternatively, the first minus sign could be eliminated by adopting the opposite sign convention for rational tangles, as indeed some authors do. We decided on the third option, which is to leave things as they are and move on.

Let M be a Seifert fiber space with orientable base surface F and exceptional fibers of multiplicities $q_1, \dots, q_n$ $(n \geq 1)$. Let V_i be a regular neighborhood of the ith exceptional fiber. Then $M_0 = M - \text{int} \coprod_{i=1}^n V_i \cong F_0 \times S^1$, where $F_0 = F - n$ open disks. Note that $H_1(\partial V_i)$ has basis c_i, t, represented by the corresponding boundary component of F_0 and $* \times S^1$, respectively. The meridian of V_i is $q_i c_i + p_i t$ for some p_i such that $(p_i, q_i) = 1$. We write $M = F(p_1/q_1, \dots, p_n/q_n)$.

A *Montesinos tangle* $\mathcal{M}(p_1/q_1, p_2/q_2)$, $q_1, q_2 \geq 2$, is the union of the two rational tangles $\mathcal{R}(p_1/q_1)$ and $\mathcal{R}(p_2/q_2)$ as illustrated in Figure 4.3. The double

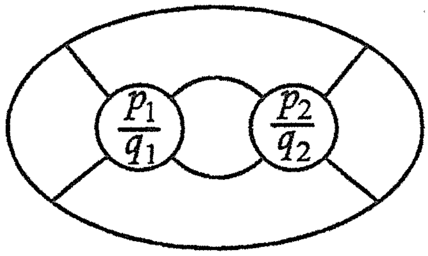


FIGURE 4.3

branched cover $\widetilde{\mathcal{M}}(p_1/q_1, p_2/q_2)$ is the Seifert fiber space $D^2(p_1/q_1, p_2/q_2)$. For example, $\widetilde{\mathcal{M}}(1/2, 1/3)$ is homeomorphic to the exterior of the trefoil.

Similarly we can consider $\mathcal{M}(p_1/q_1, \dots, p_n/q_n)$, $q_i \geq 2$, for $n \geq 3$; see Figure 4.4. Its double branched cover is the Seifert fiber space $D^2(p_1/q_1, \dots, p_n/q_n)$. Capping off the tangle as shown in Figure 4.4 we get the *Montesinos knot or link* $K[p_1/q_1, \dots, p_n/q_n]$, with double branched cover $S^2(p_1/q_1, \dots, p_n/q_n)$.

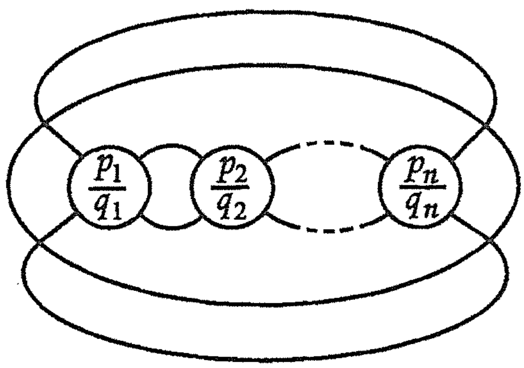


FIGURE 4.4

4.3. Tangles with non-simple double branched covers

Let $\mathcal{T} = (B, A)$ be a tangle, and let $\mathcal{F} = (F, P) \subset (B, A)$, where F is a properly embedded surface in B meeting A transversely in a finite number of points P. Let $\widetilde{\mathcal{F}} \subset \widetilde{\mathcal{T}}$ be the double branched cover of $\mathcal{F}$. Then $\widetilde{\mathcal{F}}$ will be an essential surface of non-negative Euler characteristic in $\widetilde{\mathcal{T}}$ if and only if certain conditions on $\mathcal{F}$ are satisfied. Here are the definitions.

(1) $(F, P) \cong (S^2, \textit{2 points})$ is *essential* if it is not the boundary of a *(3-ball, unknotted arc)* $\subset (B, A)$. Then $\widetilde{\mathcal{F}}$ is an essential sphere in $\widetilde{\mathcal{T}}$.
(2) $(F, P) \cong (D^2, \textit{1 point})$ is *essential* if ∂F does not bound a disk $D \subset \partial B$ such that $D \cap A$ is a single point. Then $\widetilde{\mathcal{F}}$ is an essential disk in $\widetilde{\mathcal{T}}$.
(3) F is a *Conway sphere* if $(F, P) \cong (S^2, \textit{4 points})$, $F - P$ is incompressible in $B - A$, and (F, P) is not parallel to $(S, A \cap S)$ for any boundary component S of B. Then $\widetilde{\mathcal{F}}$ is an essential torus in $\widetilde{\mathcal{T}}$.
(4) F is a *Conway disk* if $(F, P) \cong (D^2, \textit{2 points})$, $F - P$ is incompressible in $B - A$, and (F, P) is not parallel to $(D, A \cap D)$ where D is a disk $\subset \partial B$. Then $\widetilde{\mathcal{F}}$ is an essential annulus in $\widetilde{\mathcal{T}}$.

This is summarized in Table 4.1.

$\mathcal{F}$	$\widetilde{\mathcal{F}}$
essential (S^2, *2 points*)	essential S^2
essential (D^2, *1 point*)	essential D^2
Conway sphere (S^2, *4 points*)	essential T^2
Conway disk (D^2, *2 points*)	essential A^2

TABLE 4.1

For example, in the Montesinos tangle $\mathcal{M}$ shown in Figure 4.3 we can see a Conway disk separating $\mathcal{M}$ into two rational tangles. This disk lifts to an essential annulus in the double branched cover $\widetilde{\mathcal{M}}$, separating $\widetilde{\mathcal{M}}$ into two solid tori. Similarly, in a Montesinos knot or link K as in Figure 4.4 with $n \geq 4$, the boundary of (for example) the Montesinos tangle $\mathcal{M}(p_1/q_1, p_2/q_2)$ is a Conway sphere, which lifts to an essential torus in the double branched cover of K.

4.4. Example: the Whitehead link

To illustrate the theory of rational tangle fillings we consider the simplest hyperbolic link, namely the Whitehead link $L = K_0 \cup K_1$; see Figure 4.5. The link L is strongly invertible; the involution τ is rotation through π about the axis $\widetilde{C}$ shown in Figure 4.5. Let N_0, N_1 be disjoint τ-invariant regular neighborhoods of K_0, K_1 respectively; then $N_i/\tau = B_i$ is a 3-ball, $i = 0, 1$. The axis $\widetilde{C}$ maps to an unknotted circle C in the quotient $S^3/\tau \cong S^3$, and $C \cap (S^3 - \text{int}(B_0 \cup B_1))$ is a tangle $\mathcal{T}$ in $S^3 - \text{int}(B_0 \cup B_1) \cong S^2 \times I$, whose double branched cover $\widetilde{\mathcal{T}}$ is the exterior of L, $M_{\text{Wh}} = S^3 - \text{int}(N_0 \cup N_1)$.

The unknotted circle C and the 3-balls B_0 and B_1 are shown in Figure 4.6. By an isotopy this may be transformed to Figure 4.7, and thence to Figure 4.8.

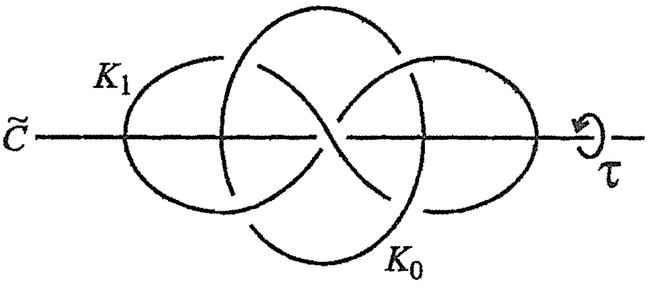


FIGURE 4.5

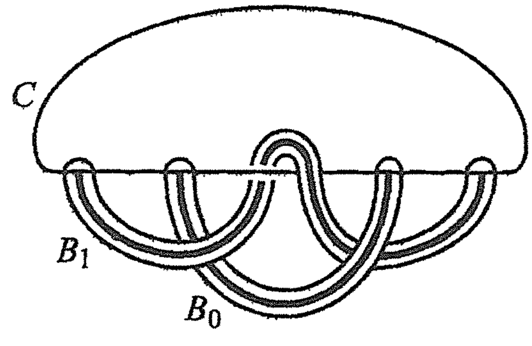


FIGURE 4.6

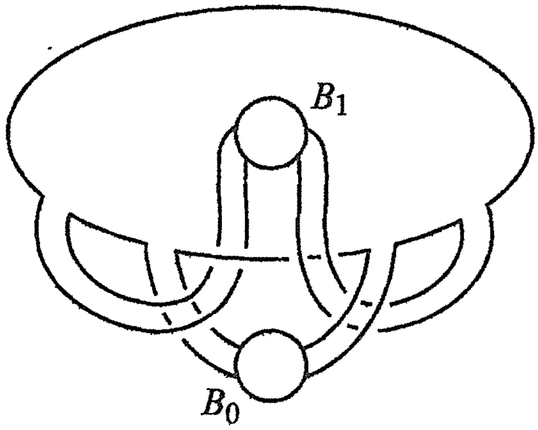


FIGURE 4.7

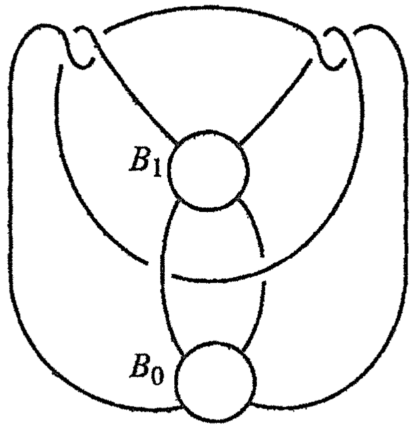


FIGURE 4.8

In getting from Figure 4.7 to Figure 4.8 the 3-ball B_0 acquires four horizontal right-handed half-twists, so if $\partial_0\mathcal{T}$ is the boundary component of the tangle $\mathcal{T}$ corresponding to B_0, then a slope $\alpha \in \mathbb{Q}\cup\{1/0\}$ with respect to the marking of $\partial_0\mathcal{T}$ determined by Figure 4.7 corresponds to the slope $\alpha-4$ with respect to the marking in Figure 4.8. Note that the former marking corresponds to the standard meridian-longitude coordinates on ∂N_0. Hence, if $\mathcal{T}(\alpha)$ denotes α-tangle filling on $\mathcal{T}$ along $\partial_0\mathcal{T}$ with respect to the marking determined by Figure 4.8, and $M_{\text{Wh}}(\alpha)$ denotes α-Dehn filling on the boundary component ∂N_0 of M_{Wh}, then by Corollary 4.4 we have

$$\widetilde{\mathcal{T}(\alpha)} \cong M_{\text{Wh}}(-\alpha-4)\ .$$

Note that since there is an isotopy of S^3 interchanging the components of L, it doesn't matter which boundary component of M_{Wh} we do the filling on.

The tangles $\mathcal{T}(\alpha)$ for $\alpha = 0, -1, -2$ and -3 are shown in Figures 4.9, 4.10, 4.11 and 4.12. Thus

$\mathcal{T}(0)$ contains a Conway sphere
$\mathcal{T}(-1) = \mathcal{M}(1/3, 1/3)$
$\mathcal{T}(-2) = \mathcal{M}(1/2, 1/4)$
$\mathcal{T}(-3) = \mathcal{M}(1/2, 1/3)$.

Moreover, since K_0 bounds a once-punctured torus in the complement of K_1, it follows easily that $M_{\text{Wh}}(0)$ contains an incompressible non-separating torus. We therefore have the following non-hyperbolic fillings on the Whitehead link exterior:

$M_{\text{Wh}}(-4)$: toroidal

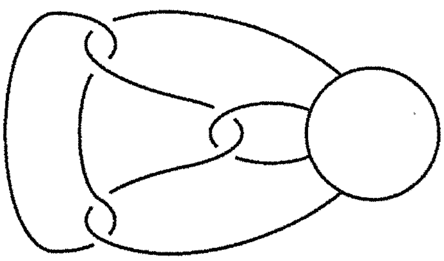


FIGURE 4.9

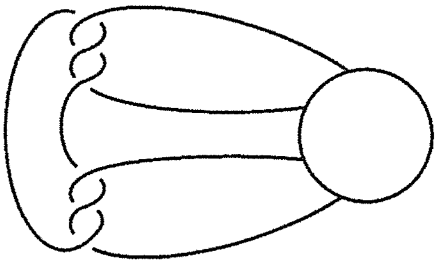


FIGURE 4.10

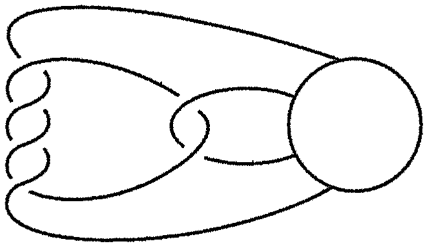


FIGURE 4.11

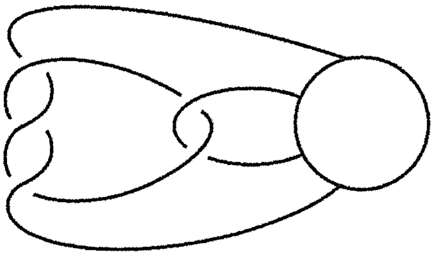


FIGURE 4.12

$M_{\mathrm{Wh}}(-3)$: SFS $D^2(3,3)$
$M_{\mathrm{Wh}}(-2)$: SFS $D^2(2,4)$
$M_{\mathrm{Wh}}(-1)$: SFS $D^2(2,3)$
$M_{\mathrm{Wh}}(0)$: toroidal.

Also $M_{\mathrm{Wh}}(1/0)$ is the exterior of K_1, which is a solid torus.

It is shown in [**NR**] that these are the only non-hyperbolic Dehn fillings on one boundary component of M_{Wh}.

The exceptional fillings on the Whitehead link exterior induce exceptional surgeries on the *twist knots*. These are the knots K_n, $n \in \mathbb{Z}$, illustrated in Figure 4.13, where there are $2n$ left-handed half-twists. (Note that one can get the correspond-

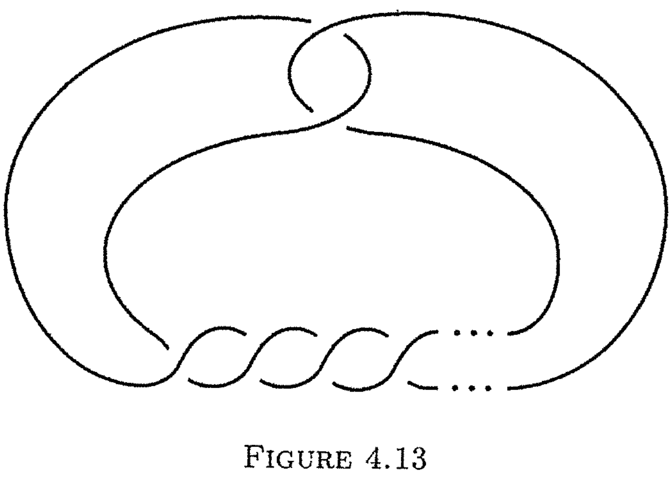


FIGURE 4.13

ing knots with an odd number of half-twists by reflecting the K_n's.) Clearly the exterior of K_n is obtained by $1/n$-filling on one boundary component of M_{Wh}. Hence K_n, $n \neq 0, -1$, is an infinite family of hyperbolic knots, each with six exceptional surgery slopes $1/0$, 0, -1, -2, -3 and -4. By [**BW**], these are the only exceptional slopes for K_n unless $n = 1$. On the other hand, K_1 = the figure-8 knot is amphicheiral, so it has ten exceptional slopes $1/0$, 0, ±1, ±2, ±3 and ±4. No other hyperbolic 3-manifold is known with more than eight exceptional slopes.

4.5. Exercises

1. Verify that the rational tangles in Figure 4.1 correspond to the rational numbers stated.
2. Let $[a_1, a_2, \ldots, a_k]$ denote the continued fraction

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}} \qquad (a_k \neq 0)$$

Define the *Euler bracket function* $e(a_1, \ldots, a_k)$ inductively as follows. Abbreviate $e(a_i, \ldots, a_j)$ by $e_{i,j}$, with the convention that $e_{1,0} = 1$ and $e_{1,-1} = 0$. Then

$$e_{1,k} = a_k e_{1,k-1} + e_{1,k-2} \, .$$

Show that

(a) $[a_1, a_2, \ldots, a_k] = \dfrac{e_{1,k}}{e_{2,k}}$

(b) $\displaystyle\prod_{i=1}^{k} \begin{bmatrix} 0 & 1 \\ 1 & a_i \end{bmatrix} = \begin{bmatrix} e_{2,k-1} & e_{2,k} \\ e_{1,k-1} & e_{1,k} \end{bmatrix}$

(c) $\displaystyle\prod_{i=1}^{k} \begin{bmatrix} a_i & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} e_{1,k} & e_{1,k-1} \\ e_{2,k} & e_{2,k-1} \end{bmatrix}$

3. Show that the result of rotating the rational tangle $\mathcal{R}(p/q)$ through $\pi/2$ about the axis perpendicular to the plane containing the four points Q is $\mathcal{R}(-q/p)$.
4. Show that the double branched cover of $K[p/q,\, a/b]$ is the lens space $L(qa + pb,\, -(sa + rb))$, where $ps - qr = 1$.
5. Let $\mathcal{T}$ be the tangle shown in Figure 4.8. Verify that $\mathcal{T}(0)$, $\mathcal{T}(-1)$, $\mathcal{T}(-2)$ and $\mathcal{T}(-3)$ are as shown in Figures 4.9, 4.10, 4.11 and 4.12.
6. Using Figure 4.8, identify the non-hyperbolic manifolds $K_n(\alpha)$, where K_n is a twist knot and $\alpha = 0, -1, -2, -3$ or -4. In particular, show that for the figure-8 knot K_1:
   $K_1(\pm1)$ is a SFS of type $S^2(2,3,7)$;
   $K_1(\pm2)$ is a SFS of type $S^2(2,4,5)$; and
   $K_1(\pm3)$ is a SFS of type $S^2(3,3,4)$.

Examples of exceptional Dehn fillings

5.1. Some examples

In this lecture we'll see some more examples of exceptional Dehn fillings constructed by means of rational tangle surgery.

We start with the tangle $\mathcal{Q}$ in the 4-punctured sphere illustrated in Figure 5.1. (As unmarked tangles, this is the same as the tangle $\mathcal{Q}$ in [**GL3**].)

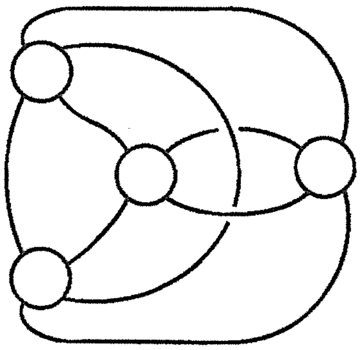


FIGURE 5.1

First recall that in Lecture 4 we showed that the Whitehead link exterior M_{Wh} is the double branched cover of the tangle illustrated in Figure 4.8. If we use the marking of $\mathcal{Q}$ determined by Figure 5.1 and order its boundary components as in the discussion immediately below, this is the tangle $\mathcal{Q}(1/2,\ 1/2,\ *, *)$. We will see in this lecture and the next that $\mathcal{Q}$ is the source of many other examples of manifolds with exceptional Dehn fillings.

Let $\mathcal{B} = \mathcal{B}(\alpha, \beta, \gamma) = \mathcal{Q}(\alpha, \beta, \gamma, *)$, a tangle in B^3 (see Figure 5.2); these are the tangles considered by Eudave-Muñoz in [**E2**]. Note that $\partial\widetilde{\mathcal{B}}$ is a single torus. It can be shown that, except for some small values of α, β and γ, $\widetilde{\mathcal{B}}$ is simple (see [**E2**] for details).

We will now see that $\mathcal{B}$ has several rational tangle fillings such that the corresponding Dehn fillings on $\widetilde{\mathcal{B}}$ are non-hyperbolic. For example,

$$\mathcal{B}(1/0) = \mathcal{M}(-1/\alpha,\ -1/\beta) \cup_\partial \mathcal{M}(-1/2,\ \gamma),$$

a union of two Montesinos tangles; see Figure 5.3.

Hence, if $1/\alpha$, $1/\beta$ and γ are non-integral then $\widetilde{\mathcal{B}}(1/0)$ is the union of two Seifert fiber spaces over the disk with two exceptional fibers:

$$\widetilde{\mathcal{B}}(1/0) = D^2(q_1, q_2) \cup_\partial D^2(2, q_3)\ ,$$

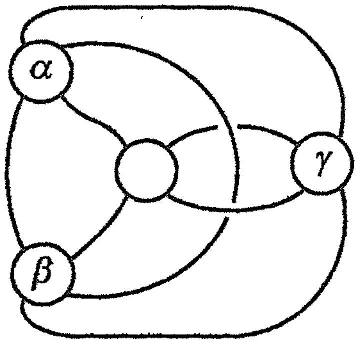


FIGURE 5.2

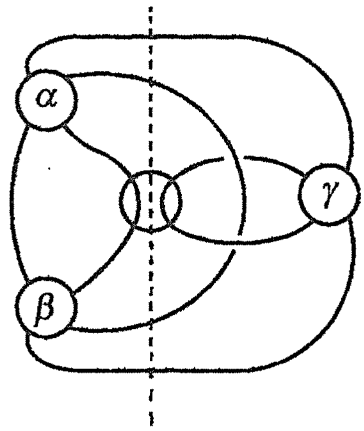


FIGURE 5.3

where q_1, q_2, q_3 are the denominators of $1/\alpha$, $1/\beta$ and γ respectively. In particular, $\widetilde{\mathcal{B}}(1/0)$ is toroidal.

Now consider $\mathcal{B}(0)$; see Figure 5.4. This is the Montesinos knot

$$K[-1/(\alpha-1), -1/(\beta-1), -1/\gamma] .$$

Its double branched cover $\widetilde{\mathcal{B}}(0)$ is therefore a Seifert fiber space over S^2 with at most three exceptional fibers.

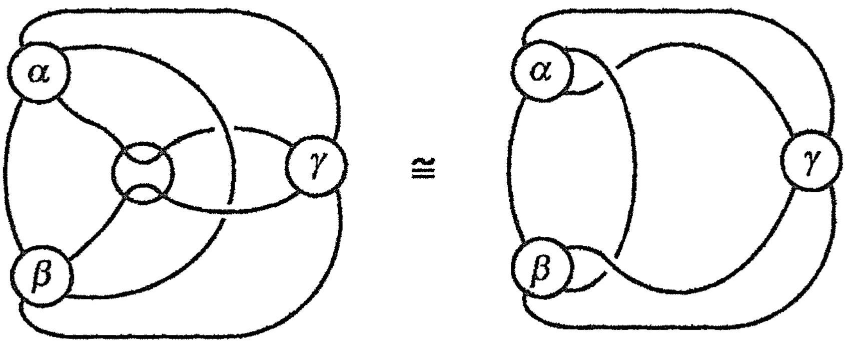


FIGURE 5.4

Similarly, $\mathcal{B}(-1)$ is the Montesinos knot

$$K[-1/(\alpha+1), -1/(\beta+1), -1/(\gamma-1)] ;$$

see Figure 5.5. So its double branched cover $\widetilde{\mathcal{B}}(1)$ is again a Seifert fiber space over S^2 with at most three exceptional fibers.

Now consider $\mathcal{B}(-1/2)$; see Figure 5.6. This is a union of two Montesinos tangles. For certain values of α, β, γ these will degenerate to rational tangles, so that $\widetilde{\mathcal{B}}(1/2)$ will be the union of two solid tori.

Furthermore, in some cases these solid tori will be glued so as to give S^3. It is a matter of arithmetic to work out when this happens (see **[E2]** for details). We then get

Theorem 5.1 (Eudave-Muñoz **[E2]**). *There are infinitely many triples (α, β, γ) such that $\widetilde{\mathcal{B}}(\alpha, \beta, \gamma)$ is simple and $\widetilde{\mathcal{B}}(\alpha, \beta, \gamma)(1/2)$ is S^3.*

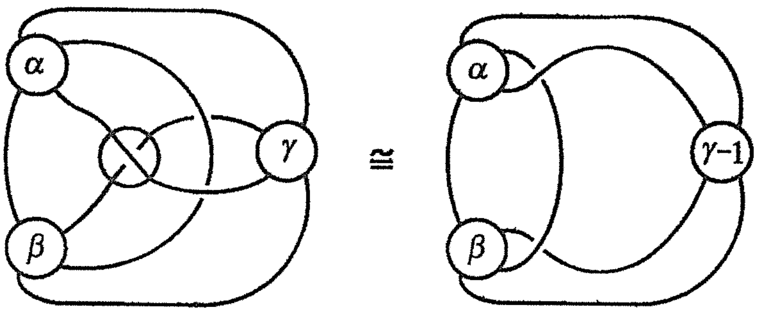


FIGURE 5.5

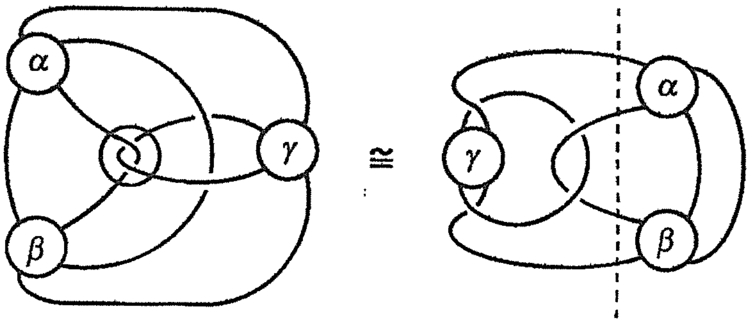


FIGURE 5.6

So for each such triple (α, β, γ), the double branched cover $\widetilde{\mathcal{B}}(\alpha, \beta, \gamma)$ is the exterior of a hyperbolic knot $E(\alpha, \beta, \gamma)$ in S^3; we call these *Eudave-Muñoz knots*, and the corresponding tangles $\mathcal{B}(\alpha, \beta, \gamma)$ *Eudave-Muñoz tangles.*

Recalling that $\widetilde{\mathcal{B}}(\alpha, \beta, \gamma)(1/0)$ is toroidal, and that $\Delta(1/2, 1/0) = 2$, we get

Corollary 5.2 (Eudave-Muñoz [**E2**]). *There are infinitely many hyperbolic knots K in S^3 such that $K(m/2)$ is toroidal for some m.*

It turns out that the Eudave-Muñoz knots are the only hyperbolic knots in S^3 with non-integral toroidal surgeries.

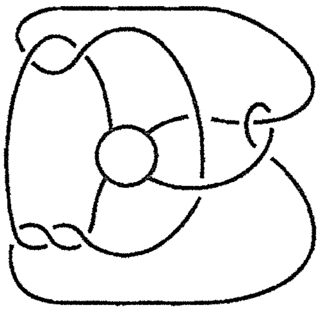
Theorem 5.3 (Gordon-Luecke [**GL3**]). *If K is a hyperbolic knot in S^3 with a non-integral toroidal surgery then K is an Eudave-Muñoz knot and the surgery is the corresponding half-integral surgery.*

The triples (α, β, γ) as in Theorem 5.1 actually fall into two infinite families, each parametrized by three integers. The corresponding tangles $\mathcal{B}(\alpha, \beta, \gamma)$ are the tangles $B(\ell, m, n, 0)$ and $B(\ell, m, 0, p)$ respectively of [**E2**].

Example

A particularly interesting special case is the Eudave-Muñoz knot $E(2, -3, 2/3)$, whose exterior is the double branched cover of the tangle $\mathcal{B}(2, -3, 2/3) = \mathcal{A}$, say, shown in Figure 5.7. As well as the exceptional tangle fillings $\mathcal{A}(-1/2)$, $\mathcal{A}(1/0)$, $\mathcal{A}(0)$ and $\mathcal{A}(-1)$ noted above, there are three others: $\mathcal{A}(-1/3)$ is a Montesinos tangle of length 3, while $\mathcal{A}(-2/3)$ and $\mathcal{A}(-2/5)$ have Conway spheres.

The knot $E(2, -3, 2/3)$ is actually the reflection of the $(-2, 3, 7)$ pretzel knot K, and the correspondence between the tangle slopes listed above and the slopes



$\mathcal{B}(2,-3,2/3)$

FIGURE 5.7

on K with respect to the usual meridian-longitude coordinates is

$$(-1/2,\ 1/0,\ 0,\ -1,\ -1/3,\ -2/3,\ -2/5) \longleftrightarrow (1/0,\ 37/2,\ 18,\ 19,\ 17,\ 20,\ 16)\ .$$

The corresponding surgeries on K are

$K(1/0) : S^3$
$K(16)$: toroidal
$K(17)$: SFS $S^2(2,3,5)$
$K(18) : L(18,5)$
$K(37/2)$: toroidal
$K(19) : L(19,7)$
$K(20)$: toroidal.

The knot K thus represents what are conjecturally all the possible types of exceptional surgeries: a lens space, a SFS of the form $S^2(q_1, q_2, q_3)$, an integral toroidal surgery, and a half-integral toroidal surgery. Indeed it has two lens space surgeries, the maximum possible by the Cyclic Surgery Theorem [**CGLS**].

Here are some additional facts about exceptional surgeries on knots.

(1) The figure-8 knot and the $(-2,3,7)$ pretzel knot are the only hyperbolic knots known with more than six exceptional surgeries, having ten and seven respectively (here we include $K(1/0) = S^3$ as exceptional).

(2) There are infinitely many hyperbolic knots with six exceptional surgeries, for example the twist knots K_n ($n \neq 0, \pm 1$) (see Section 4.4). Infinitely many Eudave-Muñoz knots also have six exceptional surgeries; see [**E2**].

(3) There are infinitely many Eudave-Muñoz knots with two lens space surgeries, for instance the family $k(2,2,n,0)$ in [**E2**].

(4) The figure-8 and $(-2,3,7)$ pretzel knots have three toroidal surgeries, $\{-4,0,4\}$ and $\{16, 37/2, 20\}$ respectively. It can be shown, using [**GW3**], that these are the only hyperbolic knots with two distinct pairs of toroidal surgeries at distance ≥ 4.

Teragaito [**Te2**] has constructed infinitely many hyperbolic knots with three toroidal surgeries of the form m, $m+1$, $m+2$ ($m \in \mathbb{Z}$).

No hyperbolic knot is known with more than three toroidal surgeries.

5.2. Chain links

Q belongs to a very interesting hierarchy of tangles, that are derived from the *chain links*. Here we will start with the *4-chain link*, which is illustrated in Figure 5.8

and which we shall denote by $4CL$. It can also be described as the $(2,2,2,-2)$

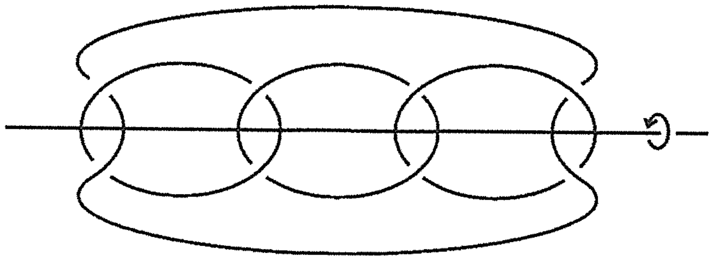


FIGURE 5.8

pretzel link, or the Montesinos link $K[1/2, 1/2, 1/2, -1/2]$. This link is strongly invertible; the involution τ is indicated in Figure 5.8, with $\mathrm{Fix}(\tau)$ an unknotted circle. The quotient of the exterior M_{4CL} is the tangle shown in Figures 5.9 and 5.10; it turns out that this is the tangle $\mathcal{Q}$.

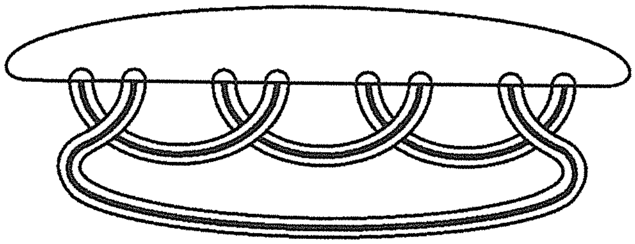


FIGURE 5.9

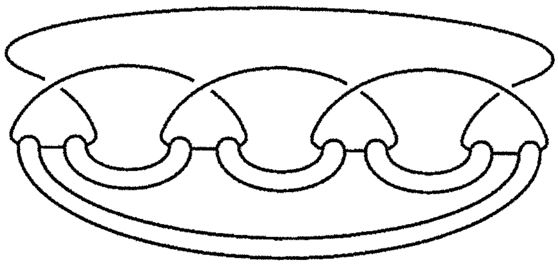


FIGURE 5.10

Let $3CL$ be the *3-chain link* shown in Figure 5.11. The quotient of the strong inversion on $3CL$ is the tangle $\mathcal{N}$ shown in Figure 5.12. Clearly (-1)-surgery on the rightmost component of $4CL$ gives $3CL$, and so there is a corresponding rational tangle filling on $\mathcal{Q}$ giving $\mathcal{N}$.

Many of the hyperbolic manifolds with exceptional fillings that are obtained by filling on M_{4CL} factor through M_{3CL}. The large number of such manifolds coming from M_{3CL} led to its being called the *magic manifold* in **[GW2]**. A complete analysis of the fillings on M_{3CL} has been given by Martelli and Petronio **[MP]**.

Also in this hierarchy of chain links is the *minimally twisted 5-chain link* $5CL$, which is illustrated in Figure 5.13 and which we shall have occasion to refer to in Section 6.3. The quotient of the strong inversion on $5CL$ is the *pentangle* $\mathcal{P}$ shown in Figure 5.14 (terminology due to John Conway). $\mathcal{P}$ is the 1-skeleton of a

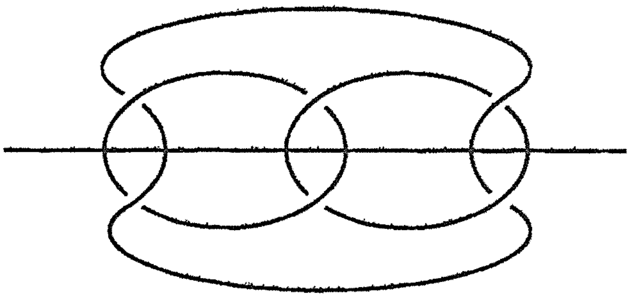


FIGURE 5.11

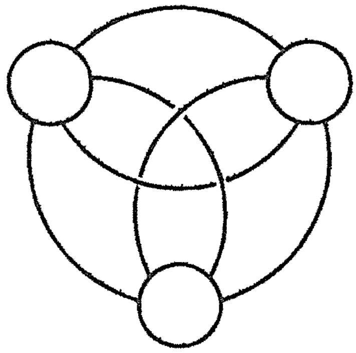


FIGURE 5.12

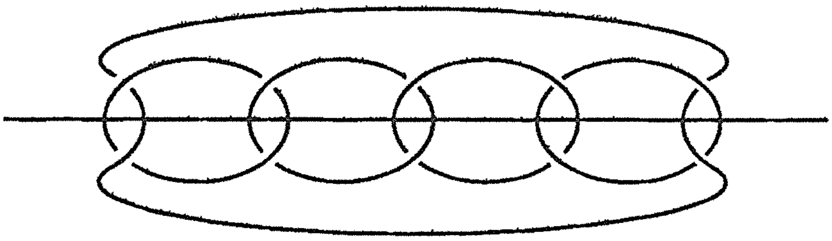


FIGURE 5.13

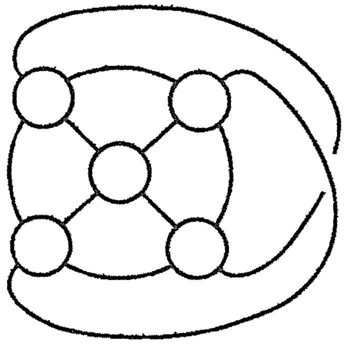


FIGURE 5.14

tetrahedron in S^3 with neighborhoods of the vertices removed. Again (-1)-surgery on the component of $5CL$ that's third from the left gives $4CL$, and so there is a corresponding filling on $\mathcal{P}$ giving $\mathcal{Q}$.

Baker has shown that the Berge knots other than those of types VII and VIII (see Section 3.5) can all be obtained by suitable Dehn surgery on four components of $5CL$ [**Ba2**]. On the other hand, the knots of types VII and VIII (those that lie on the fiber of the trefoil or figure-8 knot) have unbounded volume, so their exteriors cannot come from Dehn filling on any fixed manifold [**Ba1**].

5.3. Simplicity of the double branched cover

If we use tangles to construct examples of hyperbolic 3-manifolds M with exceptional fillings as double branched covers, the question arises as to how to show that M is hyperbolic. This is usually quite straightforward.

Recall that M is hyperbolic if and only if it is simple. So if M is not hyperbolic then it either contains an essential sphere or torus F, or is a SFS. Now such an

F will remain essential in "most" Dehn fillings $M(\gamma)$ on M, while if M is a SFS then $M(\gamma)$ is either a SFS or reducible. In the situation we're interested in we will have already constructed M so that it has two non-hyperbolic fillings $M(\alpha)$ and $M(\beta)$. By examining these, possibly together with one or two additional non-hyperbolic fillings, and using the observation above about a generic filling $M(\gamma)$, one can usually argue that M can neither contain an essential sphere or torus nor be a SFS.

5.4. Exercises

1. Let $\mathcal{A} = \mathcal{B}(2, -3, 2/3)$. Verify that
   (a) $\mathcal{A}(-1/2)$ is the unknot;
   (b) $\mathcal{A}(-1/3)$ is a Montesinos tangle of length 3;
   (c) $\mathcal{A}(-2/3)$ and $\mathcal{A}(-2/5)$ contain Conway spheres.
2. Show that the tangle shown in Figure 5.10 is the tangle $\mathcal{Q}$.

Classification of some exceptional fillings

6.1. Some classification theorems

It is natural to divide the problem of classifying all exceptional pairs $(M; \alpha_1, \alpha_2)$ into several cases, where we consider particular classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of non-hyperbolic 3-manifolds (for example, reducible manifolds, lens spaces, and so on) and assume that $M(\alpha_i) \in \mathcal{C}_i$, $i = 1, 2$. Each case can then be approached in three stages:

(A) find the smallest constant $\Delta_0 = \Delta_0(\mathcal{C}_1, \mathcal{C}_2)$ such that if $(M; \alpha_1, \alpha_2)$ is exceptional, with $M(\alpha_i) \in \mathcal{C}_i$, $i = 1, 2$, then $\Delta(\alpha_1, \alpha_2) \leq \Delta_0$;
(B) determine all such $(M; \alpha_1, \alpha_2)$ with $\Delta(\alpha_1, \alpha_2) = \Delta_0$;
(C) determine all such $(M; \alpha_1, \alpha_2)$ with $\Delta(\alpha_1, \alpha_2) < \Delta_0$.

(A) has been solved in many cases, and even (B) is known for several pairs of classes $\mathcal{C}_1, \mathcal{C}_2$. As $\Delta(\alpha_1, \alpha_2)$ gets smaller for a given pair $\mathcal{C}_1, \mathcal{C}_2$, more examples tend to occur and the classification problem becomes harder, so it is probably too optimistic to expect a complete solution to (C) in general.

Table 6.1 shows the values of $\Delta_0(\mathcal{C}_1, \mathcal{C}_2)$ for pairs of the following classes of 3-manifolds: those that contain an essential S^2, D^2, A^2 or T^2, $\{S^3\}$, and {lens spaces}. (The entries marked $*$ obviously do not occur.) In particular, in all cases except (S, S^3) and (T, L), Δ_0 has been determined. Regarding the unknown cases, note that (S, S^3) probably never happens (as mentioned in Lecture 2, this is equivalent to the Cabling Conjecture), while it is known that $\Delta_0(T, L)$ is either 3 or 4 [**G3**], [**L3**]. For references for the entries in Table 6.1, see [**G3**].

Table 6.2 shows the status of (B) for the same classes.

	S	D	A	T	S^3	L
S	1	0	2	3	?	1
D		1	2	2	$*$	$*$
A			5	5	$*$	$*$
T				8	2	?
S^3					0	1
L						1

TABLE 6.1

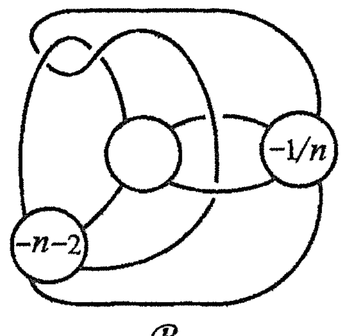
	S	D	A	T	S^3	L
S	?	✓	✓	✓	?	?
D		?	?	✓	*	*
A			✓	✓	*	*
T				✓	✓	?
S^3					✓	?
L						?

TABLE 6.2

Let us describe the entries in Table 6.2 in more detail.

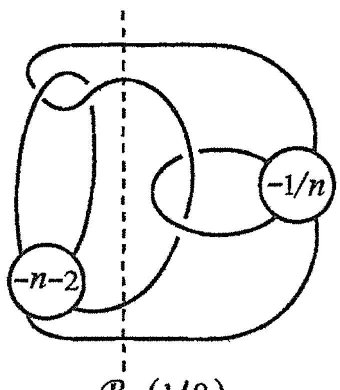
For (T, S^3) we have Theorem 5.3 in Lecture 5: the manifolds M are precisely the exteriors of the Eudave-Muñoz knots.

For (S, T), where $\Delta_0 = 3$, we have the following examples of Eudave-Muñoz and Wu [**EW**], again coming from the tangles $\mathcal{B}(\alpha, \beta, \gamma)$. Specifically, let $\mathcal{B}_n = \mathcal{B}(2, -n-2, -1/n)$, $n \geq 2$, as shown in Figure 6.1. Then $\widetilde{\mathcal{B}}_n(1/0)$ is toroidal; the



$\mathcal{B}_n$

FIGURE 6.1



$\mathcal{B}_n(1/0)$

FIGURE 6.2

corresponding Conway sphere in $\mathcal{B}_n(1/0)$ is shown in Figure 6.2. (Of course this is just a special case of Figure 5.3.)

Also, Figure 6.3 shows that $\mathcal{B}_n(-1/3)$ is the connected sum of a trefoil and the Hopf link, hence $\widetilde{\mathcal{B}}_n(1/3) \cong L(3,1) \# \mathbb{R}P^3$. Finally, it can be shown that $\widetilde{\mathcal{B}}_n$ is simple if $n \geq 2$. The following theorem of Kang [**K**] says that these are the only examples of a reducible filling and a toroidal filling on a hyperbolic 3-manifold at distance 3.

Theorem 6.1 (Kang [**K**]). *M is a simple 3-manifold with $M(\alpha_1)$ reducible, $M(\alpha_2)$ toroidal and $\Delta(\alpha_1, \alpha_2) = 3$ if and only if $(M; \alpha_1, \alpha_2) \cong (\widetilde{\mathcal{B}}_n;\ 1/3,\ 1/0)$ for some $n \geq 2$.*

We now describe some fillings that contain essential disks and essential annuli; here we need manifolds with at least two boundary components. Let $\mathcal{A}(\alpha, \beta)$ be the tangle in $S^2 \times I$ shown in Figure 6.4. Note that $\mathcal{A}(\alpha, \beta)$ is obtained by removing the γ-tangle from $\mathcal{B}(\alpha, \beta, \gamma)$. Also the boundary of $\widetilde{\mathcal{A}}(\alpha, \beta)$ consists of two tori.

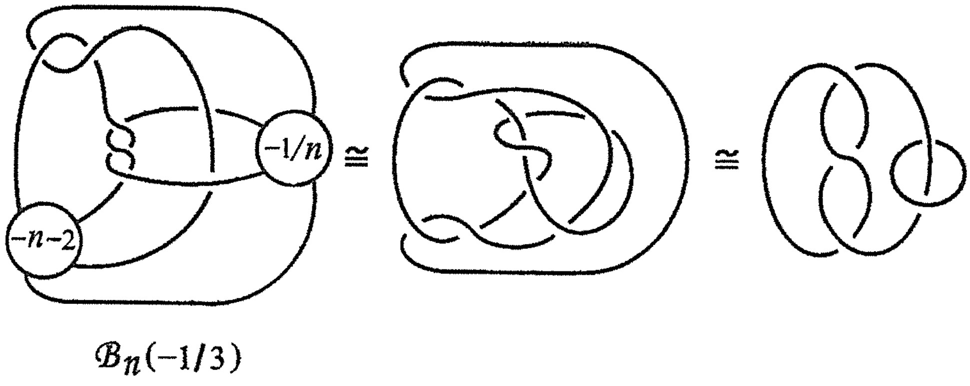


FIGURE 6.3

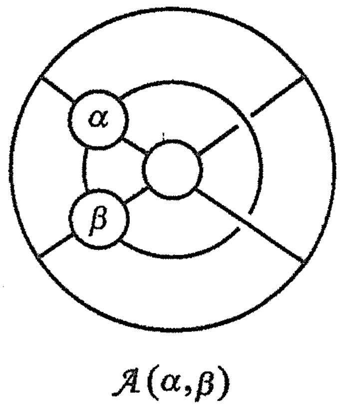


FIGURE 6.4

Let $\mathcal{A}(\alpha,\beta)(\rho)$ denote ρ-filling on the inner boundary component of $\mathcal{A}(\alpha,\beta)$, with respect to the obvious marking. Then $\mathcal{A}(\alpha,\beta)(-1/2)$ contains a Conway sphere and a Conway disk (provided neither α nor β is integral), as shown in Figure 6.5. It follows that $\widetilde{\mathcal{A}}(\alpha,\beta)(1/2)$ is toroidal and annular.

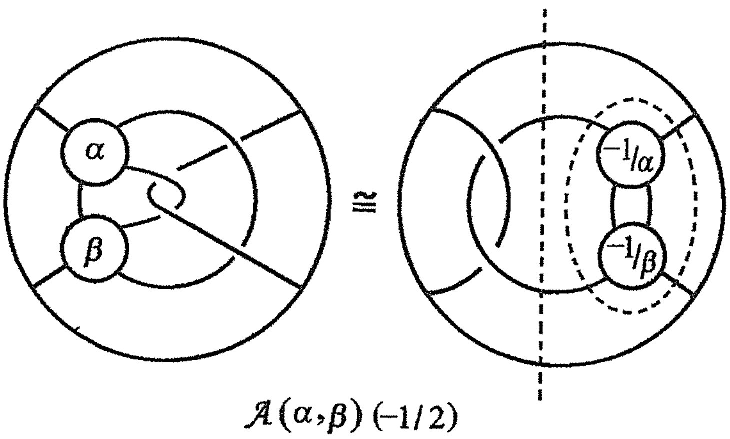
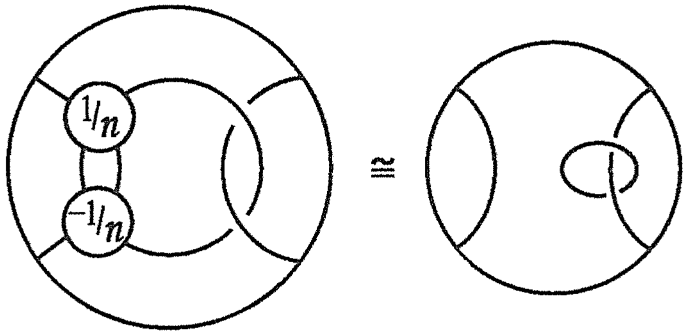


FIGURE 6.5

For a second interesting filling, we specialize to $\mathcal{A}_n = \mathcal{A}(1/n, -1/n)$. Then $\mathcal{A}_n(1/0)$ is as shown in Figure 6.6. Taking double branched covers gives $\widetilde{\mathcal{A}}_n(1/0) \cong \mathbb{R}P^3 \# S^1 \times D^2$. Hence $\widetilde{\mathcal{A}}_n(1/0)$ is reducible and ∂-reducible.



$\mathcal{A}_n(1/0)$

FIGURE 6.6

The following theorem of Lee [L1] says that these examples account for all reducible and annular fillings at distance 2.

Theorem 6.2 (Lee [**L1**]). *M is a simple 3-manifold with $M(\alpha_1)$ reducible, $M(\alpha_2)$ annular and $\Delta(\alpha_1, \alpha_2) = 2$ if and only if $(M; \alpha_1, \alpha_2) \cong (\widetilde{\mathcal{A}}_n;\ 1/0,\ 1/2)$ for some $n \geq 3$.*

As noted above, $\widetilde{\mathcal{A}}_n(1/0)$ is ∂-reducible and $\widetilde{\mathcal{A}}_n(1/2)$ is toroidal. However, there are other examples of ∂-reducible and toroidal fillings at distance 2, closely related to the Eudave-Muñoz knots. Recall that the exteriors of these knots are the double branched covers of the tangles $\mathcal{B}(\alpha, \beta, \gamma)$ for two infinite families of (α, β, γ). Removing the γ-tangle from the first family and the β-tangle from the second family gives two families of tangles in $S^2 \times I$, of the form $\mathcal{A}(\alpha, \beta)$ and $\mathcal{A}'(\alpha, \gamma)$, say. Call the union of the two families $\mathbb{E}$. If $\mathcal{T} \in \mathbb{E}$, denote by $\mathcal{T}(\rho)$ the ρ-filling on the boundary component of $\mathcal{T}$ that corresponds to the boundary of $\mathcal{B}(\alpha, \beta, \gamma)$. Then it turns out that $\widetilde{\mathcal{T}}(1/2) \cong S^1 \times D^2$ and $\widetilde{\mathcal{T}}(1/0)$ is toroidal.

Theorem 6.3 (Gordon-Luecke [**GL3**]). *M is a simple 3-manifold with $M(\alpha_1)$ a solid torus, $M(\alpha_2)$ toroidal and $\Delta(\alpha_1, \alpha_2) = 2$ if and only if $(M; \alpha_1, \alpha_2) \cong (\widetilde{\mathcal{T}};\ 1/2,\ 1/0)$ for some $\mathcal{T} \in \mathbb{E}$.*

Regarding the more general situation of ∂-reducible and toroidal fillings at distance 2, Lee [**L2**] has shown the following.

Theorem 6.4 (Lee [**L2**]). *M is a simple 3-manifold with $M(\alpha_1)$ ∂-reducible, $M(\alpha_2)$ toroidal and $\Delta(\alpha_1, \alpha_2) = 2$ if and only if $(M; \alpha_1, \alpha_2) \cong$ either $(\widetilde{\mathcal{A}}_n;\ 1/0,\ 1/2)$ for some $n \geq 3$, or $(\widetilde{\mathcal{T}};\ 1/2,\ 1/0)$ for some $\mathcal{T} \in \mathbb{E}$.*

Let's move on to the (T, T) entry in Table 6.2. Recall that $\Delta_0(T, T) = 8$. This is realized by the figure-8 knot K: we saw in Lecture 3 that $K(4) \cong K(-4)$ is toroidal. It turns out that there is another, closely related simple manifold with two toroidal fillings at distance 8, with slopes β_1, β_2, say, the *figure-8 sister manifold* $M'_{\text{fig } 8}$.

Theorem 6.5 (Gordon [**G1**]). *M is a simple 3-manifold with $M(\alpha_1)$ and $M(\alpha_2)$ toroidal and $\Delta(\alpha_1, \alpha_2) = 8$, if and only if $(M; \alpha_1, \alpha_2) \cong$ either $(M_{\text{fig } 8}; 4, -4)$ or $(M'_{\text{fig } 8}; \beta_1, \beta_2)$.*

Here, and in the sequel, to avoid overburdening the notation we write $(M; \alpha_1, \alpha_2) \cong (N; \alpha'_1, \alpha'_2)$ to mean that there is a homeomorphism from M to N taking (α_1, α_2) to either (α'_1, α'_2) or (α'_2, α'_1).

We remark that both the figure-8 exterior and the figure-8 sister manifold are fillings on one component of the Whitehead link: $M_{\text{fig } 8} \cong M_{\text{Wh}}(1)$ and $M'_{\text{fig } 8} \cong M_{\text{Wh}}(-5)$.

Regarding the cases (A, A) and (A, T), the relevant example here is the $(-2, 3, 8)$ pretzel link L, also known as the *Whitehead sister* link. It turns out that there are two fillings on the exterior M_{Whsis} of L, with slopes γ_1, γ_2, say, at distance 5, each of which is both annular and toroidal. (Since there is an involution of M_{Whsis} interchanging its boundary components, it doesn't matter which boundary component we do the fillings on. With respect to the usual meridian-longitude coordinates on the trefoil component of L, $\{\gamma_1, \gamma_2\} = \{9, 13/2\}$; see **[GW3]**.)

There is a nice tangle surgery description of this. Namely, M_{Whsis} is the double branched cover of the tangle $\mathcal{A}(-3, 2)$, and the two annular and toroidal fillings correspond to $\mathcal{A}(-3, 2)(1/0)$ and $\mathcal{A}(-3, 2)(-2/5)$. See **[GW1]** for more details.

Theorem 6.6 (Gordon-Wu **[GW1]**, **[GW2]**). *M is a simple 3-manifold with $M(\alpha_1)$ annular, $M(\alpha_2)$ either annular or toroidal, and $\Delta(\alpha_1, \alpha_2) = 5$, if and only if $(M; \alpha_1, \alpha_2) \cong (M_{\text{Whsis}}; \gamma_1, \gamma_2)$.*

We now make some remarks on the unknown entries in Table 6.2.

$\underline{(S, S)}$: Several examples of pairs of reducible fillings at distance 1 on simple 3-manifolds are known **[GLi]**, **[EW]**, **[HM]**, but it is not clear what the general picture should be. The examples of Eudave-Muñoz and Wu **[EW]** actually have two torus boundary components; maybe one could at least show that these are the only examples with more than one boundary component. Hoffman and Matignon **[HM]** raise the interesting question as to whether one (or both) reducible fillings always has an $L(2, 1)$, $L(3, 1)$ or $L(4, 1)$ summand.

$\underline{(S^3, L)}$: This is the Berge Conjecture (see Lecture 3).

$\underline{(S, L)}$: Some examples are known, but again, as in the (S, S) case, the general picture is not clear.

$\underline{(L, L)}$: There is an obvious analog of the Berge Conjecture here, but this does not appear to have been investigated.

$\underline{(D, D)}$: If $M(\alpha_1)$ and $M(\alpha_2)$ are ∂-reducible, $\alpha_1 \neq \alpha_2$, then by **[W4]** $\partial M - T_0$ has exactly one component. When this component is a torus, then $M(\alpha_1)$ and $M(\alpha_2)$ are solid tori (by **[Sc]**) and the corresponding manifolds M have been classified, as discussed in Section 3.5. The general case is still not completely understood; see **[W1]**, **[W2]**.

$\underline{(D, A)}$: If a 3-manifold whose boundary consists of at least two tori is ∂-reducible then it is also reducible. So Theorem 6.2 also gives all simple manifolds whose boundary consists of tori with a ∂-reducible and annular filling at distance 2. However, Frigerio, Martelli and Petronio **[FMP]** have constructed, for any $g \geq 2$, simple 3-manifolds M with ∂-reducible and annular fillings at distance 2 where ∂M consists of a torus and a surface of genus g.

As noted earlier, the case (S, S^3) probably never occurs, and for (T, L), we don't even know what Δ_0 is.

Finally we come to (C). The only cases where the classification has been carried out for any values of $\Delta = \Delta(\alpha_1, \alpha_2)$ strictly less than $\Delta_0(\mathcal{C}_1, \mathcal{C}_2)$ are (A, A), (A, T) and (T, T).

The first two have been done for $\Delta = 4$. The relevant examples here are the exteriors M_{Wh} and $M_{10/3}$ of the Whitehead link and the rational link $K[10/3]$ respectively. (We observe that the Whitehead link is the rational link $K[8/3]$.) Each of these has two fillings at distance 4, both of which are annular and toroidal. With respect to the usual meridian-longitude coordinates, the filling slopes are $0, -4$ (see Section 4.4) and $-2, 2$ (see [**GW3**]), respectively. (Since in each case there is an involution of S^3 interchanging the components of the link, it doesn't matter which boundary component we do the filling on.)

Theorem 6.7 (Gordon-Wu [**GW1**], [**GW2**]). *M is a simple 3-manifold with $M(\alpha_1)$ annular, $M(\alpha_2)$ either annular or toroidal, and $\Delta(\alpha_1, \alpha_2) = 4$, if and only if $(M; \alpha_1, \alpha_2) \cong$ either $(M_{\text{Wh}}; 0, -4)$ or $(M_{10/3}; -2, 2)$.*

The (T, T) case, where $\Delta_0 = 8$, has been done for all $\Delta \geq 4$. For $\Delta \geq 6$ this was done in [**G1**]. It turns out that for $\Delta = 7$ or 6, in each case there is exactly one example. Like the two examples realizing $\Delta = 8$ (Theorem 6.5), these are also fillings on one boundary component of M_{Wh}, namely $M_{\text{Wh}}(-5/2)$ and $M_{\text{Wh}}(2)$ respectively. Let the corresponding toroidal filling slopes be δ_1, δ_2 and $\varepsilon_1, \varepsilon_2$.

Theorem 6.8 (Gordon [**G1**]). *M is a simple 3-manifold with $M(\alpha_1)$ and $M(\alpha_2)$ toroidal, where $\Delta(\alpha_1, \alpha_2) = 7$ or 6, if and only if $(M; \alpha_1, \alpha_2) \cong$ either $(M_{\text{Wh}}(-5/2); \delta_1, \delta_2)$ or $(M_{\text{Wh}}(2); \varepsilon_1, \varepsilon_2)$, respectively.*

When $\Delta = 5$ a new phenomenon occurs. The four manifolds in Theorems 6.5 and 6.8 all have boundary a single torus. However, as mentioned above in the discussion preceding Theorem 6.6, M_{Whsis} has two toroidal fillings $M_{\text{Whsis}}(\gamma_1)$ and $M_{\text{Whsis}}(\gamma_2)$ at distance 5. Then, for infinitely many slopes γ on the other boundary component of M_{Whsis}, $M_{\text{Whsis}}(\gamma)$ will be simple and $M_{\text{Whsis}}(\gamma)(\gamma_i) = M_{\text{Whsis}}(\gamma_i)(\gamma)$ will be toroidal, $i = 1, 2$. So the strict finiteness in Theorems 6.5 and 6.8 no longer holds. It is shown in [**GW3**] that the examples M_{Whsis} and $M_{\text{Whsis}}(\gamma)$ together with six other manifolds with a single torus boundary component, are the only simple 3-manifolds with two toroidal fillings at distance 5. Similarly, there are three manifolds with two torus boundary components and four with a single torus boundary component that account for all toroidal fillings on simple 3-manifolds at distance 4.

Theorem 6.9 (Gordon-Wu [**GW3**]).

(1) *There exist $(M_i; \alpha_1^{(i)}, \alpha_2^{(i)})$, $1 \leq i \leq 7$, where*
   - (i) *M_i is a simple 3-manifold;*
   - (ii) *∂M_i consists of a single torus T_0, $2 \leq i \leq 7$, and two tori, T_0 and T_1, if $i = 1$;*
   - (iii) *$\alpha_1^{(i)}, \alpha_2^{(i)}$ are slopes on T_0 with $\Delta(\alpha_1^{(i)}, \alpha_2^{(i)}) = 5$;*
   - (iv) *$M_i(\alpha_1^{(i)})$ and $M_i(\alpha_2^{(i)})$ are toroidal;*
   - (v) *if M is a simple 3-manifold with $M(\alpha_1), M(\alpha_2)$ toroidal and $\Delta(\alpha_1, \alpha_2) = 5$, then $(M; \alpha_1, \alpha_2) \cong$ either $(M_i; \alpha_1^{(i)}, \alpha_2^{(i)})$ for some i, $1 \leq i \leq 7$, or $(M_1(\gamma); \alpha_1^{(1)}, \alpha_2^{(1)})$ for some slope γ on T_1.*

(2) *There exist $(N_i; \beta_1^{(i)}, \beta_2^{(i)})$, $1 \leq i \leq 7$, where*

(i) *N_i is a simple 3-manifold;*
(ii) *∂N_i consists of a single torus T_0; $4 \le i \le 7$, and two tori, T_0 and T_1, $1 \le i \le 3$;*
(iii) *$\beta_1^{(i)}, \beta_2^{(i)}$ are slopes on T_0 with $\Delta(\beta_1^{(i)}, \beta_2^{(i)}) = 4$;*
(iv) *$N_i(\beta_1^{(i)})$ and $N_i(\beta_2^{(i)})$ are toroidal;*
(v) *if M is a simple 3-manifold with $M(\alpha_1), M(\alpha_2)$ toroidal and $\Delta(\alpha_1, \alpha_2) = 4$, then $(M; \alpha_1, \alpha_2) \cong$ either $(N_i; \beta_1^{(i)}, \beta_2^{(i)})$ for some i, $1 \le i \le 7$, or $(N_i(\gamma); \beta_1^{(i)}, \beta_2^{(i)})$ for some i, $1 \le i \le 3$, and for some slope γ on T_1.*

As mentioned above, the manifold M_1 in part (1) of Theorem 6.9 is M_{Whsis}. The manifolds N_1 and N_2 in part (2) are M_{Wh} and $M_{10/3}$. (N_3 is not the exterior of a link in S^3.)

Theorems 6.5, 6.6, 6.7, 6.8 and 6.9, together with the other S, D, A, T entries in Table 6.1, show that these four manifolds $M_1, N_i, 1 \le i \le 3$, are the only simple manifolds with more than one boundary component having a pair of non-simple fillings at distance > 3.

Corollary 6.10 (Gordon-Wu [**GW3**]). *Let M be a simple 3-manifold with a torus boundary component T_0 and at least one other boundary component, and let α_1, α_2 be slopes on T_0 such that $M(\alpha_1)$ and $M(\alpha_2)$ are not simple. Then either $\Delta(\alpha_1, \alpha_2) \le 3$ or $(M; \alpha_1, \alpha_2) \cong (M_1; \alpha_1^{(1)}, \alpha_2^{(1)})$ or $(N_i; \beta_1^{(i)}, \beta_2^{(i)}), 1 \le i \le 3$.*

6.2. Seifert fiber spaces

The one class of non-hyperbolic manifolds that is noticeably missing from Tables 6.1 and 6.2 is the class $\mathcal{S}$ of Seifert fiber spaces of type $S^2(q_1, q_2, q_3)$. These have proved to be the hardest to analyze in this context, and in particular the best possible bounds $\Delta_0(\mathcal{S}, \mathcal{C})$, where $\mathcal{C} = \mathcal{S}, S, T, \{S^3\}$ or {lens spaces}, have not yet been established. (See [**BCSZ1**] and [**BCSZ2**] for results on $\Delta_0(\mathcal{S}, S)$.)

If we restrict to those manifolds in $\mathcal{S}$ with finite fundamental group, i.e. where $\{q_1, q_2, q_3\} = \{2, 2, n\}$, $\{2, 3, 3\}$, $\{2, 3, 4\}$ or $\{2, 3, 5\}$, then considerably more is known. For example, using F to denote this class, we have the following extension to Table 6.1:

	S	T	S^3	L	F
F	1	?	?	2	3

TABLE 6.3

The bounds in cases (F, L) and (F, F) were established in [**BZ1**] and [**BZ3**] respectively. In [**BCSZ2**] the bound of 1 for $\Delta_0(F, S)$ was obtained except in a special case, which was done in [**BGZ**]. Finally, regarding Conjecture 3.3, it is known that $\Delta_0(F, S^3) \le 2$, i.e. a surgery on a hyperbolic knot that yields a manifold with finite fundamental group is either integral or half-integral [**BZ1**].

6.3. Methods of proof; non-integral toroidal surgeries

We conclude these lectures with a brief description of how the classification theorems discussed above are proved, focusing on Theorem 5.3, which describes the hyperbolic knots with non-integral toroidal surgeries. In particular this will show how the theory of tangle fillings enters into the picture.

The method we will describe starts with a hyperbolic 3-manifold M, with two Dehn fillings $M(\alpha_1)$ and $M(\alpha_2)$ containing "interesting" surfaces S_1 and S_2 respectively. In practice, "interesting" means that the surface is either essential or a Heegaard surface (i.e. it splits the manifold into two handlebodies). For example, if $M(\alpha)$ is non-simple then by definition it contains an essential sphere, disk, annulus or torus, while M itself contains no such surface. Moreover, recall from Section 1.1 that if $M(\alpha)$ is non-hyperbolic then it is either non-simple or a closed small SFS. (Incidentally, one of the reasons that relatively little is known about the SFS's of type $S^2(q_1, q_2, q_3)$ is that the smallest interesting surface they contain is a genus 2 Heegaard surface, while manifolds in the classes listed in Table 6.1 all contain essential or Heegaard surfaces of genus at most 1.)

We have $M(\alpha_i) = M \cup V_i$, where V_i is a solid torus whose meridian has slope α_i on the torus boundary component T_0 of M, $i = 1, 2$. We may assume that the surface S_i meets V_i in n_i meridian disks, giving rise to a punctured surface $F_i = S_i \cap M$ in M such that $F_i \cap T_0$ consists of n_i curves of slope α_i, $i = 1, 2$. We may also isotop F_1 and F_2 so that they intersect in a disjoint union of arcs and simple closed curves, and so that each component of $F_1 \cap T_0$ meets each component of $F_2 \cap T_0$ in $\Delta = \Delta(\alpha_1, \alpha_2)$ points.

To get useful information from this set-up we need some non-triviality condition on F_1 and F_2, and it turns out that it is enough to have first, that $n_i > 0$, $i = 1, 2$, and second, that no arc component of $F_1 \cap F_2$ is boundary parallel in F_1 or F_2. These conditions in turn are guaranteed by our assumptions on S_1 and S_2: if both S_1 and S_2 are essential and cannot be moved into M, then standard topological arguments enable us to choose F_1 and F_2 suitably, while if one or both of the S_i's is a Heegaard surface, the conditions can be achieved by putting the core of V_i in *thin position* [**Ga2**] with respect to S_i.

One regards $F_1 \cap F_2$ as defining graphs Γ_i in S_i, $i = 1, 2$, where the "fat" vertices of Γ_i are the meridian disks $S_i \cap V_i$, and the edges of Γ_i are the arc components of $F_1 \cap F_2$. Thus there is a natural bijection between the edges of Γ_1 and the edges of Γ_2. Note that each vertex of Γ_i has valency Δn_j ($\{i, j\} = \{1, 2\}$). The idea now is to study the combinatorics of such pairs of intersection graphs, using the faces of Γ_i to get topological information about the pair $(M(\alpha_j), S_j)$. This technique for studying Dehn surgery was introduced by Litherland [**Li**], and has been considerably developed over the last 25 years.

The first thing one shows is that, for a pair of surfaces (S_1, S_2) of a given type, there is an upper bound on Δ. (It's not hard to see that this should be the case: as Δ gets larger, so does the valency of the vertices of Γ_i, and so for S_i of a fixed topological type, Γ_i will have to accumulate more and more topologically parallel edges. For Δ large enough one can piece together the bigon faces associated with families of parallel edges in both graphs to construct an essential annulus in M, contradicting our hypothesis.) One hopes to sufficiently refine the combinatorial/topological analysis of the pair of intersection graphs (Γ_1, Γ_2) to eventually get the best possible upper bound Δ_0 on Δ for the given type of pair of surfaces (S_1, S_2). Many of the entries in Table 6.1 are obtained in this way.

Once Δ_0 has been established, one then analyzes the situation at the critical value $\Delta = \Delta_0$. What tends to happen here is that one of the numbers of punctures (vertices) n_1 or n_2 has to be small. One then tries to understand the possible pairs of intersection graphs in enough detail that the possible triples $(M; F_1, F_2)$ can be explicitly determined. This is how the entries in Table 6.2 are obtained. In some cases the analysis can be extended to determine all possible triples $(M; F_1, F_2)$ for values of Δ strictly less than Δ_0.

We illustrate this general approach with a slightly more detailed discussion of the proof of Theorem 5.3. Let K be a hyperbolic knot in S^3, with exterior M. Let μ be the slope on ∂M of the meridian of K, and suppose ν is a slope such that $M(\nu)$ is toroidal and $\Delta = \Delta(\nu, \mu) \geq 2$. So here $\alpha_1 = \mu$, $\alpha_2 = \nu$, and we take S_1 to be a Heegaard sphere $S \subset S^3 = M(\mu)$, and S_2 to be an incompressible torus $T \subset M(\nu) = M \cup V$, say. We put K in thin position with respect to S, and assume that the number of components of $T \cap V$ is minimal over all incompressible tori in $M(\nu)$. We then get a punctured sphere $P = S \cap M$ and a punctured torus $F = T \cap M$ in M, whose intersection defines graphs $\Gamma_S \subset S$, $\Gamma_T \subset T$ as above. Write t for n_2, the number of boundary components of F.

The first step is to use the main combinatorial result of [**GL2**], which says that because the graph Γ_T *does not represent all types*, the graph Γ_S contains a special kind of subgraph Λ (a *great web* in the terminology of [**GL3**]). One of the important features of Λ is that its vertices correspond to points of intersection of K with S of the same sign. An analysis of the faces of Λ of order 2 and 3, and their topological implications for the pair $(M(\nu), T)$, lead to the conclusion that $\Delta = 2$ and $t = 2$ or 4 [**GL3**]. The case $t = 4$ is ruled out in [**GL4**] by a detailed examination of the faces of Λ of order ≤ 4.

So now Γ_T is a graph in the torus T with two vertices, and $\Delta = 2$. The proof of Theorem 5.3 is completed as follows; for details see [**GL5**]. Since the vertices of Λ all have same sign, an edge of Λ corresponds to an edge of Γ_T that joins the two vertices. The latter edges belong to at most four parallelism classes on T, and we label each edge of Λ with the class of the corresponding edge of Γ_T. We say that a (disk) face f of Λ is *good* if the edges of ∂f have exactly two labels, and one of these labels has the property that no two consecutive edges in ∂f have that label. Let $M(\nu) = X_1 \cup_T X_2$; thus the faces of Γ_S lie alternately in X_1 or X_2. A topological argument shows that the existence of a good face that is contained in X_i implies that X_i is a SFS of type $D^2(q_1, q_2)$. On the other hand it is shown that Λ contains a good face f_i in X_i for $i = 1$ and 2. This is proved by index arguments on the dual graph Λ^* of Λ with various edge orientations determined by the labeling of the edges of Λ. We conclude that $M(\nu) = X_1 \cup_T X_2$ is the union of two SFS's of type $D^2(q_1, q_2)$ along their boundary. (See Figure 6.7 for a schematic depiction.) Moreover, it turns out that the Seifert fibers of X_1 and X_2 intersect once on T.

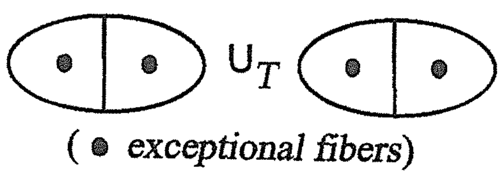


FIGURE 6.7

It follows that if we remove from $M(\nu)$ regular neighborhoods of the exceptional fibers in X_1 and X_2 we get the exterior of the "doubled" Hopf link L_0 shown in Figure 6.8.

Let $K_0 \subset M(\nu)$ be the core of the filling solid torus V. A detailed analysis of the situation, using the good faces f_1 and f_2, shows that in the complement of the four exceptional fibers, K_0 appears as in Figure 6.9. Let L be the 5-component

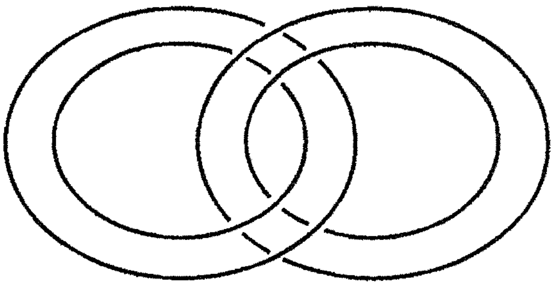


FIGURE 6.8

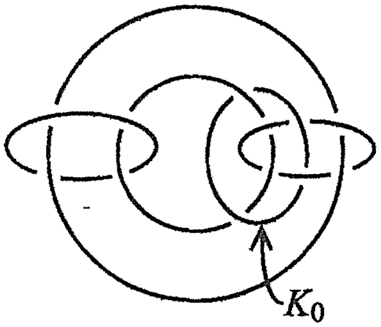


FIGURE 6.9

link $L_0 \cup K_0$. Then M can be obtained by Dehn filling on M_L, the exterior of L, along the four boundary components corresponding to the 4-component sublink L_0. It turns out that M_L is homeomorphic to the exterior M_{5CL} of the minimally twisted 5-chain link discussed in Lecture 5. Recall that the quotient of M_{5CL} under the strong inversion on $5CL$ is the pentangle $\mathcal{P}$; see Figure 5.14. Hence M is the double branched cover of some tangle $\mathcal{T}$ of the form $\mathcal{P}(\alpha, \beta, \gamma, \delta, *)$, where $\alpha, \beta, \gamma, \delta$ correspond to the four exceptional fibers.

We now note that (i) some filling on $\mathcal{T}$ is the unknot, namely that corresponding to the filling $M(\mu) \cong S^3$, and (ii) for $\chi \in \{\alpha, \beta, \gamma, \delta\}$, $\Delta(\chi, \lambda) \geq 2$ where λ is the longitude of the corresponding component of L_0, since $\Delta(\chi, \lambda)$ is the multiplicity of the corresponding exceptional fiber. Using these observations, together with known facts about the distances between various exceptional Dehn fillings, we show that $1/2 \in \{\alpha, \beta, \gamma, \delta\}$. By symmetry, we may assume $\delta = 1/2$. Since $\mathcal{P}(*, *, *, 1/2, *) = \mathcal{Q}$ (as unmarked tangles), we conclude that $\mathcal{T} = \mathcal{Q}(\alpha', \beta', \gamma', *) = \mathcal{B}(\alpha', \beta', \gamma')$, where α' etc. are the slopes corresponding to α etc. under the change of marking. Finally, one identifies the filling slopes on $\mathcal{B}(\alpha', \beta', \gamma')$ that give the unknot (corresponding to $M(\mu) \cong S^3$) and the union of two Montesinos tangles (corresponding to $M(\nu) = X_1 \cup_T X_2$): with respect to the marking in Figure 5.2 they turn out to be $-1/2$ and $1/0$ respectively. Thus $\mathcal{B}(\alpha', \beta', \gamma')(-1/2)$ is the unknot, and so $\mathcal{B}(\alpha', \beta', \gamma')$ is an Eudave-Muñoz tangle, K is an Eudave-Muñoz knot $E(\alpha', \beta', \gamma')$, and the toroidal surgery on K is the half-integral surgery described in Section 5.1.

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