Connections, Curvature + Characteristic Classes

from Appendix C of Milnor, Characteristic Classes.

Note by Conan Leung

§ Cpx. VB: $\mathbb{C}^n \to E \to M$

Connection $\nabla : \underbrace{\Gamma(M,E)}_{\Omega^0(M,E)} \longrightarrow \underbrace{\Gamma(M, T^*_{\mathbb{C}} \otimes_{\mathbb{C}} E)}_{\Omega^1(M,E)}$ $\mathbb{C}$-linear

s.t. $\nabla(fs) = df \otimes s + f \nabla s$ $\quad f : M \to \mathbb{C}$, $s \in \Gamma(M,E)$.

- If
$$\begin{array}{ccc} E|_U & \cong & U \times \mathbb{C}^n \\ \downarrow \uparrow s_i & & \downarrow \nearrow (0,\dots,\overset{i\text{th}}{1},\dots 0) \\ M \supset U & = & U \end{array}$$
write $\nabla s_i = \sum_j \omega_i^j \otimes s_j$

$\exists\, \omega_i^j \in \Omega^1(U,\mathbb{C})$

then $\forall\, s = \sum_i f^i s_i \in \Gamma(E|_U)$ w/ $f^i : U \to \mathbb{C}$

$\nabla s = \sum df^i \otimes s_i + f^i \nabla s_i = \sum (df^i \delta_i^j + \omega_i^j f^i) s_j$

$= (d + A)\begin{pmatrix} f^1 \\ \vdots \\ f^n \end{pmatrix}$ w/ $A = (\omega_i^j) \in \underbrace{\Omega^1(U, \mathrm{End}\,\mathbb{C}^n)}_{\Omega^1(U, \mathrm{End}\,E)}$

- ∇_0, ∇_1 connections $\Longrightarrow \nabla_t := t\nabla_1 + \underbrace{(1-t)\nabla_0}$ conn.

$= \nabla_0 + t(\nabla_1 - \nabla_0)$

$\in \Omega^1(M, \mathrm{End}\,E)$

(t can be any function on M)

$\mathcal{A}(E) := \{ \text{conn. on } E \}$

$= \nabla_0 + \Omega^1(M, \mathrm{End}\,E)$ (if nonempty)

Namely, $\mathcal{A}(E)$ is affine space modelled on vector space $\Omega^1(M, \mathrm{End}\,E)$

- $\mathcal{A}(E)$ non-empty

$\left(\begin{array}{l} \nabla = d \text{ on } E|_{U_\alpha} \simeq U_\alpha \times \mathbb{C}^n \\ \rho_\alpha\text{'s partit}^{n} \text{ of 1 on cover } U_\alpha\text{'s} \end{array} \Longrightarrow \sum \rho_\alpha \nabla_\alpha \text{ conn. on whole } E. \right)$

- $\exists!$ $\mathbb{C}$-linear $\nabla : \Omega^1(M,E) \longrightarrow \Omega^2(M,E)$

  s.t. $\nabla(\underset{\Omega^1(M),\ \Gamma(E)}{\theta \otimes s}) = d\theta \otimes s - \theta \wedge \overset{\text{old.}}{\nabla s}$

Similarly, $\nabla : \Omega^k(M,E) \longrightarrow \Omega^{k+1}(M,E)$

Lemma. $\nabla^2(fs) = f\,\nabla^2 s$ i.e $C^\infty(M,\mathbb{C})$-linear

i.e. $\exists$ $F_\nabla \in \Omega^2(M, \mathrm{End}\,E)$ s.t. $\nabla^2 s = F_\nabla \cdot s$

curvature.

Pf: $\nabla^2(fs) = \nabla(df \cdot s + f\,\nabla s)$

$= (\underbrace{d(df)}_{d^2=0} s - \cancel{df \cdot \nabla s}) + \cancel{df \wedge \nabla s} + f\,\nabla^2 s$

Explicitly, $E|_U \overset{\text{loc.}}{\simeq} U \times \mathbb{C}^n$

$\nabla = d + (\omega_i^j)_{n\times n}$ w/ $\omega_i^j \in \Omega^1(U)$

then $F_\nabla = d\omega_i^j - \sum_k \omega_k^j \wedge \omega_i^k =: \Omega_i^j$

- Fact (Integrability). $F_\nabla = 0$ (flat connection)

  $\Rightarrow$ loc. $E|_{U_\alpha} = U_\alpha \times \mathbb{C}^n$ s.t. $\nabla|_{U_\alpha} = d$

  $\Rightarrow$ gluing functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n,\mathbb{C})$

  s.t. $d\,g_{\alpha\beta} = 0$ (i.e. loc. const. fu.)

- ∇_j on E_j $\rightsquigarrow$ ∇ on $E_1 \otimes E_2$ $\left(\nabla(s_1 \otimes s_2) = (\nabla_1 s_1) \otimes s_2 + s_1 \otimes \nabla_2 s_2\right)$

- ∇ on E $\rightsquigarrow$ ∇^* on E^* s.t. $E^* \otimes E \xrightarrow{\mathrm{Tr}} \mathbb{C}$

  $\nabla^* \otimes \nabla$ & d compatible.

Bianchi Identity $\nabla F_\nabla = 0 \in \Omega^3(M, \mathrm{End}\,E)$

Pf: $\nabla F_\nabla \overset{\text{loc.}}{=\!=} d\Omega - \omega \wedge \Omega + \Omega \wedge \omega$

$= d(\underbrace{\cancel{d}\omega}_{d^2=0} - \omega^2) - \omega\,(d\omega - \underbrace{\omega^2}) + (d\omega - \underbrace{\omega^2})\,\omega$

(and $d(\omega^2) = (d\omega)\wedge\omega - \omega \wedge (d\omega)$)

- $d(\mathrm{Tr}\, F_\nabla) = \mathrm{Tr}(\nabla F_\nabla) = 0 \in \Omega^3(M)$
- $d(\mathrm{Tr}\, F_\nabla^2) = 0$
- Lemma. (1) $\mathrm{Tr}(F_\nabla)^k \in \Omega^{2k}(M)$

  s.t. $d(\text{— " —}) = 0$

  (2) $[\mathrm{Tr}(F_\nabla)^k] \in H^{2k}(M)$

  indep. of choice of ∇

Pf. of (2). $\nabla_0,\ \nabla_1 = \nabla_0 + A$ conn. on E/M

$\rightsquigarrow$

$$\begin{tikzcd} \pi^*E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d] \\ M\times[0,1]_t \arrow[r, "\pi"] & M \end{tikzcd}$$

Conn. on π^*E

$$\tilde{\nabla} := \pi^*\nabla_0 + tA$$

$$= \frac{d}{dt}\otimes dt + \underbrace{\nabla_0 + tA}_{\nabla_t},$$

$$\Rightarrow \quad F_{\tilde\nabla} = F_{\nabla_t} + A\, dt$$

$$\Rightarrow \quad \mathrm{Tr}\, F_{\tilde\nabla}^k \Big|_{t=0}^{t=1} = \int_{t=0}^{t=1} \frac{d}{dt}(\mathrm{Tr}\, F_{\tilde\nabla}^k)\, dt$$

$$\mathrm{Tr}\, F_{\nabla_1}^k - \mathrm{Tr}\, F_{\nabla_0}^k = \int \mathrm{Tr}\,(\underset{\text{Bianchi}}{\cancel{\tilde\nabla}} - \nabla_t)(F_{\nabla_t} + A\,dt)^k$$

$$= \int - d\, k\, \mathrm{Tr}\, F_{\nabla_t}^{k-1} A\ dt$$

$$= d\left(-k \int_{t=0}^{t=1} \mathrm{Tr}\, F_{\nabla_t}^{k-1} A\, dt\right).$$

Def. $\mathrm{ch}_k(E) := \left[\mathrm{Tr}\, \frac{\left(\frac{i}{2\pi}F_\nabla\right)^k}{k!}\right] \in H^{2k}(M, \mathbb{Q})$

or $\mathrm{ch}(E) = \sum_k \mathrm{ch}_k(E) = \left[\mathrm{Tr}\, e^{\frac{i}{2\pi}F_\nabla}\right]$

Chern character.

Note: $\mathrm{ch}_0(E) = \mathrm{rank}(E)$; $\mathrm{ch}_1(E) = \frac{i}{2\pi}[\mathrm{Tr}\, F_\nabla]$

- $ch(E_1 \oplus E_2) = ch(E_1) + ch(E_2)$

  $\left(\because\ F_{E_1 \oplus E_2} = \begin{pmatrix} F_{E_1} & \\ & F_{E_2} \end{pmatrix}\right)$ (from conn. on E_i's.)

- $ch(E_1 \otimes E_2) = ch(E_1) \cdot ch(E_2)$

  $\left(\because\ F_{E_1 \otimes E_2} = F_{E_1} \otimes I_{E_2} + I_{E_1} \otimes F_{E_2}\right)$

$\Rightarrow ch : K(M) \longrightarrow H^{ev}(M, \mathbb{Q})$ ring homo. $\left(\begin{matrix}\text{Fact:} \\ \text{isom}/\mathbb{Q}\end{matrix}\right)$

Can replace $\mathrm{Tr}(-)^k : \mathrm{End}\, \mathbb{C}^n \longrightarrow \mathbb{C}$ by any inv. polyn. $P(gXg^{-1}) = P(X)$, $\forall g \in GL(n, \mathbb{C})$.

Eg. $P(X) = \det(I + tX)$

$\rightsquigarrow$ Chern class $c(E) = [P(\frac{i}{2\pi} F_A)] \in H^{ev}(M, \mathbb{Z})/\mathrm{Tor}$.

$ch_1 = c_1$, $ch_2 = (c_1^2 - 2c_2)/2$ etc.

$c(E_1 \oplus E_2) = c(E_1) \cdot c(E_2)$.

§ Real vector bundle $\mathbb{R}^n \longrightarrow E \longrightarrow M$

Assume $\langle\ ,\ \rangle : E \otimes E \longrightarrow \mathbb{R}$ pos. def. inner product

∇ compatible with $\langle\ ,\ \rangle$

$\overset{\text{def.}}{\Longleftrightarrow} \forall\ s, s' \in \Gamma(M, E)$, $d\langle s, s'\rangle = \langle \nabla s, s'\rangle + \langle s, \nabla s'\rangle$

$\Longleftrightarrow \nabla \langle\ \ \rangle = 0$ by viewing $\langle\ \ \rangle \in \Gamma(M, E^* \otimes E^*)$.

wrt local orthonormal frame $s_1, \dots, s_n$ of E

$\Longleftrightarrow \nabla = d + (\omega_i^j)$ satisfies $\omega_i^j + \omega_j^i = 0$.

∇ torsion free when $E = T^*_M$

$$\overset{\text{def.}}{\Longleftrightarrow}\quad \begin{tikzcd} \Omega^0(T^*_M) \arrow[r, "\nabla"] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & \Omega^1(T^*_M) = \Gamma(T^*_M \otimes T^*_M) \arrow[d, "\wedge"] \\ \Omega^1 \arrow[r, "d"] & \Omega^2 = \Gamma(T^*_M \wedge T^*_M) \end{tikzcd}$$

$\Longleftrightarrow$ $\nabla(dx^k) = \sum \Gamma^k_{ij}\, dx^i \otimes dx^j$ then $\Gamma^k_{ij} = \Gamma^k_{ji}$

$\Longleftrightarrow$ $\forall$ fu. f, $\nabla(df) \in \Gamma(\mathrm{Sym}^2\, T^*_M)$ (called Hessian).

- Fundamental thm. in Riemannian Geometry:
  $\forall$ (M, g), $\exists!$ ∇ on T^*_M s.t. torsion-free + compat. w/ g (called Levi-Civita connection).

§ Chern's Gauss-Bonnet Theorem

(M^2, g) oriented.

$T^*_M|_U$ w/ local o.n. basis θ_1, θ_2

$\rightsquigarrow$ $\nabla = d + \begin{pmatrix} 0 & \omega_{12} \\ -\omega_{12} & 0 \end{pmatrix}$ $\qquad \omega_{12} \in \Omega^1(U)$

$\rightsquigarrow$ $F_\nabla = \begin{pmatrix} 0 & \Omega_{12} \\ -\Omega_{12} & 0 \end{pmatrix}$ $\qquad \Omega_{12} \in \Omega^2(\underset{\sim\sim}{M})$

Write $\Omega_{12} = \underbrace{K}_{\text{Gauss curvature.}}\, dA$ $\qquad$ ($dA = -\theta_1 \wedge \theta_2$)

Gauss-Bonnet Thm.

$$\frac{1}{2\pi}\int_M \Omega_{12} = \chi(M) \qquad \text{if } M \text{ closed}$$

Pf: $\forall$ oriented $\mathbb{R}^2 \longrightarrow (E, g) \longrightarrow M$ $\rightsquigarrow$ cpx. str. (on fibers)

($\because$ $SO(2) = U(1)$, ie. $J =$)

Connection $\begin{pmatrix} & \omega_{12} \\ -\omega_{12} & \end{pmatrix} \longleftrightarrow i\omega_{12}$

$[i\Omega_{12}] \overset{(\text{Trace for rk } 1)}{=\!=\!=}$ char. class $=\!=$ $c \cdot e(E)$ $\exists c$

Check S^2 e.g. $\Rightarrow$ $C = 2\pi i$ QED.

• $\forall\, V\ (\simeq \mathbb{R}^{2n})$ w/ metric $g \in S^2 V^*$ & vol. $\nu \in \Lambda^{max} V^*$
w/ $|\nu|_g = 1$ ($\pm\nu$ unique)

$$\begin{tikzcd}
\langle A-,-\rangle \in \Lambda^2 V^* \arrow[r, "\Lambda^n(-)"] & \Lambda^{2n} V^* \\
A \in \underline{o}(V,g) \arrow[u, "\sim"'] \arrow[r, bend left, "\mathrm{Pfaff}"] & \mathbb{R} \arrow[u, "\sim", "\nu"']
\end{tikzcd}$$

$o(2n) = \left\{ \begin{matrix} \text{skew-symm.} \\ \text{matrices} \end{matrix} \right\}$

• $\mathrm{Pfaff}\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = a$

• $(\mathrm{Pfaff}(A))^2 = \det(A)$

• $\mathrm{Pfaff}(BAB^t) = \mathrm{Pfaff}(A) \cdot \det B$

• oriented $\mathbb{R}^{2n} \longrightarrow (E, g) \longrightarrow M$

wrt local oriented orthonormal frame

$\Longrightarrow$ $\Omega = (\Omega^j_i) \in \Omega^2(U, \underline{o}(2n))$

$\rightsquigarrow$ $\mathrm{Pfaff}(\Omega) \in \Omega^{2n}$ (global defined)

• $d(\text{—''—}) = 0$ ($\because$ Bianchi)

<u>Thm.</u> $[\mathrm{Pfaff}(\Omega)]/(2\pi)^n = e(E) \in H^{2n}(M, \mathbb{R})$.

Pf: Similar reason. Take E to be universal in dim. $\leq 4n$. $(\mathrm{Pf}(\Omega))^2 = \det(\Omega) = p_n(E)$ (upto $(2\pi)^{\bullet}$)
$\Longrightarrow$ $\mathrm{Pf}(\Omega) = \pm\, e(E)$. $+ \leftarrow \because \checkmark$ for $\mathbb{R}^2$-bdl.