MATH 5012

Exercise 7

The following two problems are for those love functional analysis.

① Every bounded set in a reflexive Banach space is weakly sequentially compact.

Let $\{x_n\}$ be bdd sequence in the space X. We'd like to find a subsequence x_{n_j}, $x_{n_j} \rightharpoonup x^*$ some $x^* \in X$, ie, $\Lambda x_{n_j} \to \Lambda x^*$, $\forall \Lambda \in X'$. Let $Y = \overline{\langle x_n \rangle}$ the separable subspace spanned by $\{x_n\}$, it is again reflexive (closed subsp. of reflexive space is reflexive). As $(Y')' = Y'' = Y$, so Y' is also separable (the dual of a normed space is separable $\Rightarrow$ normed space itself separable.) Let $\{\Lambda_m\}$ be a dense subset.

Then use Cantor diagonal argument to pick $\{x_{n_k}\}$ s.t.

$\lim_{k\to\infty} \Lambda_m x_{n_k}$ exists for each m. Argue that $\forall \Lambda \in Y'$

$\lim_{k\to\infty} \Lambda x_{n_k}$ also exists. Define $y(\Lambda) = \lim_{k\to\infty} \Lambda x_{n_k}$, $y \in (Y')'$.

and use reflexivity to get x^*.

② Helly's Theorem: Let X be a separable Banach space. Then every bounded sequence $\{\Lambda_n\}$ in X' contains a weakly* convergent subsequence, ie, $\exists \Lambda_{n_j}$ and $\Lambda \in X'$ s.t.

$$\Lambda_{n_j} x \to \Lambda x, \quad \forall x \in X.$$

The proof is basically the same as ①.

A remark A set in a normed space is *weakly sequentially compact* if every sequence in this set contains a weakly convergent sequence. A set in a normed space is *weakly compact* if it is compact in the weak topology. When the underlying space is a metric space, a set is weakly sequentially compact iff it is weakly compact. However, an ∞-dim normed space is not metrizable in weak topology. We can't apply the general result. However, a theorem of Eberlein-Smulyan asserts that this remains true for Banach spaces. We try not to use this heavy tool.

Whether a bounded set in a Banach space is weakly sequential compact (or weakly compact) we have the following statement: yes if and only if the space is reflexive. ① establishes one direction, and the other direction is more difficult.

③ In the proof of Slicing Measures, we use

$$\int g(x)\,d\mu(x,y) = \int g(x)\,d\sigma(x), \quad \forall g \in C_b(\mathbb{R}^n)$$

in Step 4. Prove it.

④ Verify that

$$x \longmapsto \chi_A(x, f(x)),$$

where f is $\mathcal{L}^n$-measurable and A $\mathcal{L}^n$-measurable, is $\mathcal{L}^n$-measurable. We have used it in Step 1 of the pf of Young Measures.

⑤ In the setting of Young measure, show that if $f_k \to f$ a.e. then $\nu_x = \delta_{f(x)}$ a.e.