Tutorial 9

March 30, 2017

1. **Expansion method**

   Solve the following inhomogeneous wave equations:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(0,t) = h(t), \qquad u(l,t) = k(t) \\ u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x) \end{cases} \tag{1}$$

Solution: First, expand every term in the problem by Fourier sine series, that is,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi}{l}x), \quad u_n(t) = \frac{2}{l}\int_0^l u(x,t)\sin(\frac{n\pi}{l}x)dx, n = 1, 2, \cdots$$

and similarly,

$$u_{tt}(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin(\frac{n\pi}{l}x), \quad v_n(t) = \frac{2}{l}\int_0^l u_{tt}(x,t)\sin(\frac{n\pi}{l}x)dx, n = 1, 2, \cdots$$

$$u_{xx}(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin(\frac{n\pi}{l}x), \quad w_n(t) = \frac{2}{l}\int_0^l u_{xx}(x,t)\sin(\frac{n\pi}{l}x)dx, n = 1, 2, \cdots$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(\frac{n\pi}{l}x), \quad f_n(t) = \frac{2}{l}\int_0^l f(x,t)\sin(\frac{n\pi}{l}x)dx, n = 1, 2, \cdots$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l}x), \quad \phi_n = \frac{2}{l}\int_0^l \phi(x)\sin(\frac{n\pi}{l}x)dx, n = 1, 2, \cdots$$

and

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin(\frac{n\pi}{l}x), \quad \psi_n = \frac{2}{l}\int_0^l \psi(x)\sin(\frac{n\pi}{l}x)dx, n = 1, 2, \cdots .$$

Note that

$$v_n(t) = \frac{2}{l}\int_0^l u_{tt}(x,t)\sin(\frac{n\pi}{l}x)dx = u_n''(t)$$

and

$$\begin{aligned} w_n(t) = \frac{2}{l}\int_0^l u_{xx}(x,t)\sin(\frac{n\pi}{l}x)dx &= \frac{2}{l}u_x(x,t)\sin(\frac{n\pi}{l}x)\Big|_0^l - \frac{2}{l}\frac{n\pi}{l}\int_0^l u_x(x,t)\cos(\frac{n\pi}{l}x)dx \\ &= -\frac{2}{l}\frac{n\pi}{l}u(x,t)\cos(\frac{n\pi}{l}x)dx\Big|_0^l - \frac{2}{l}(\frac{n\pi}{l})^2\int_0^l u(x,t)\sin(\frac{n\pi}{l}x)dx \\ &= -\frac{2n\pi}{l^2}\{(-1)^n k(t) - h(t)\} - (\frac{n\pi}{l})^2 u_n(t). \end{aligned}$$

Then the problem implies that

$$\begin{aligned} & u_n''(t) - c^2[-\frac{2n\pi}{l^2}\{(-1)^n k(t) - h(t)\} - (\frac{n\pi}{l})^2 u_n(t)] = f_n(t) \\ & u_n(0) = \phi_n, u_n'(t) = \phi_n \end{aligned}$$

by the uniqueness of Fourier series, i.e.

$$u_n''(t) + c^2\lambda_n u_n(t) = f_n(t) - c^2\frac{2n\pi}{l^2}\{(-1)^n k(t) - h(t)\} =: g_n(t)$$
$$u_n(0) = \phi_n, u_n'(t) = \phi_n$$

where $\lambda_n = \beta_n^2 = (\frac{n\pi}{l})^2$ this is a second order inhomogeneous ODE with constant coefficients, and it is solvable. More precisely, the general solution to the corresponding homogeneous problem is

$$\bar{u}_n(t) = A_n \cos(c\beta_n t) + B_n \sin(c\beta_n t) =: A_n y_1 + B_n y_2$$

where A_n, B_n are constants to be determined and $y_1 = \cos(c\beta_n t), y_2 = \sin(c\beta_n t)$. A particular solution to the inhomogeneous problem is given by variation of parameters method

$$\tilde{u}_n(t) = -y_1(t)\int_{t_0}^{t}\frac{y_2(s)g_n(s)}{W[y_1, y_2]}ds + y_2(t)\int_{t_0}^{t}\frac{y_1(s)g_n(s)}{W[y_1, y_2]}ds$$

where the Wronskian is given by $W[y_1, y_2] = \begin{vmatrix} y_1, y_2 \\ y_1', y_2' \end{vmatrix}$. Hence the general solution to above ODE is $u_n = \bar{u}_n + \tilde{u}_n = A_n \cos(c\beta_n t) + B_n \sin(c\beta_n t) + \tilde{u}_n$, where A_n, B_n are determined by initial conditions. Therefore, the solution to (1) is $u(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi}{l}x)$.

2. **Shifting data method**

   Consider the following inhomogeneous wave equations:

$$\begin{cases} u_{tt} - c^2 u_{xx} = F(x)\cos wt \\ u(0,t) = H\cos wt, \qquad u(l,t) = K\cos wt \\ u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x) \end{cases} \tag{2}$$

   We wish to subtract a solution of

$$\begin{cases} U_{tt} - c^2 U_{xx} = F(x)\cos wt \\ U(0,t) = H\cos wt, \qquad U(l,t) = K\cos wt \end{cases}$$

   A good guess is that U should have the form $U(x,t) = u_0(x)\cos wt$, thus $u_0(x)$ satisfies

$$\begin{cases} -w^2 u_0 - c^2 u_0'' = F(x) \\ u_0(0) = H, \qquad u_0(l) = K \end{cases}$$

   This is a solvable second order ODE. Thus we can find a special solution $U(x,t) = u_0(x)\cos wt$.

   Let u be a solution to (1), set $v = u - U$, then v satisfies

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 \\ v(0,t) = 0, \qquad v(l,t) = 0 \\ v(x,0) = \phi(x) - u_0(x), \quad u_t(x,0) = \psi(x) \end{cases} \tag{3}$$

   This is a solvable homogeneous wave problem which we can use the seperation of variables, for example.

3. **Δ_3 in spherical coordinates**

   For the three-dimensional laplacian

$$\Delta_3 = \partial_x^2 + \partial_y^2 + \partial_z^2$$

   it is natural to use spherical coordinates (r, θ, ϕ). First, consider the chain of variables $(x, y, z) \to (s, \phi, z)$ which is given by

$$x = s\cos\phi$$

$$y = s\sin\phi$$

$$z = z$$

By the two-dimensional Laplace calculation, we have

$$u_{xx} + u_{yy} = u_{ss} + \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi}.$$

Second, consider the chain of variables $(s, \phi, z) \to (r, \phi, \theta)$ which is given by

$$s = r\sin\theta$$

$$z = r\cos\theta$$

$$\phi = \phi$$

By the two-dimensional Laplace calculation, we have

$$u_{ss} + u_{zz} = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Thus we have

$$\Delta_3 u = u_{xx} + u_{yy} + u_{zz} = \frac{1}{s}u_s + \frac{1}{s^2}u_{\phi\phi} + u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

And note that $s = r\sin\theta$ and $u_s = u_r\frac{\partial r}{\partial s} + u_\theta\frac{\partial\theta}{\partial s} = u_r\frac{s}{r} + u_\theta\frac{\cos\theta}{r}$. Therefore

$$\Delta_3 u = \frac{1}{r^2}\cot\theta u_\theta + \frac{1}{r^2\sin^2\theta}u_{\phi\phi} + u_{rr} + \frac{2}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$