Assignment 7 for MATH4220

April 20, 2017

Exercise 7.1: 1, 2, 3, 4, 5, 6
Extra Problems:

1. Consider the following problem
$$\begin{cases} \Delta u = u^3 \text{ in } D \\ \dfrac{\partial u}{\partial n} + a(x)u = h \text{ on } \partial D \end{cases} \tag{1}$$
where
$$a(x) \geq 0.$$
Show that the solution to (1) (if exists) is unique.

2. Consider the following problem
$$\begin{cases} \Delta u - b(x)u = f(x) \text{ in } D \\ u = h \text{ on } \partial D \end{cases} \tag{2}$$
Let u be a C^2 function.
   (a) Define an energy functional $E[u]$ associated with (2).
   (b) Show that u is a solution to (2) if and only if
$$E[w] \geq E[u] \quad \forall w \in C^2, w = h \text{ on } \partial D$$

Exercise 7.2: 2, 3
Extra Problem: Formulate and prove Exercise 7.2.2 in two-dimensional case.
Exercise 7.3: 1, 3
Exercise 7.4: 1, 2, 3, 5, 6, 7, 9, 10, 13, 15, 17(a), 19, 20, 21
Extra Problems:

3. Given $h(x), g(y)$, find a solution formula for the following problem
$$\begin{cases} \Delta u = 0 & \text{in } Q = \{x > 0, y > 0\} \\ u(x,0) = h(x), & x > 0 \\ u(0,y) = g(y), & y > 0 \end{cases} \tag{3}$$
Hint: Find the Green's function in Q first.

4. Solve the following problem
$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0 \text{ in } \{(x,y,z) | z > 0\} \\ u(x,y,0) = \dfrac{1}{\sqrt{x^2 + y^2 + 1}} \\ u \text{ is bounded.} \end{cases} \tag{4}$$
What happens if the condition that u is bounded is dropped?

Exercise 7.1

1. Derive the three-dimensional maximum principle from the mean value property.

2. Prove the uniqueness up to constants of the Neumann problem using the energy method.

3. Prove the uniqueness of the Robin problem $\partial u/\partial n + a(\mathbf{x})u(\mathbf{x}) = h(\mathbf{x})$ provided that $a(\mathbf{x}) > 0$ on the boundary.

4. Generalize the energy method to prove uniqueness for the diffusion equation with Dirichlet boundary conditions in three dimensions.

5. Prove Dirichlet's principle for the Neumann boundary condition. It asserts that among *all* real-valued functions $w(\mathbf{x})$ on D the quantity

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 d\mathbf{x} - \iint_{\text{bdy } D} hw\ dS$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \text{ in } D, \qquad \frac{\partial u}{\partial n} = h(\mathbf{x}) \text{ on bdy } D.$$

It is required to assume that the average of the given function $h(\mathbf{x})$ is zero (by Exercise 6.1.11).

Notice three features of this principle:

(i) There is *no constraint at all* on the trial functions $w(\mathbf{x})$.

(ii) The function $h(\mathbf{x})$ appears in the energy.

(iii) The functional $E[w]$ does not change if a constant is added to $w(\mathbf{x})$.

(*Hint:* Follow the method in Section 7.1.)

6. Let A and B be two disjoint bounded spatial domains, and let D be their exterior. So bdy $D =$ bdy $A \cup$ bdy B. Consider a harmonic function $u(\mathbf{x})$ in D that tends to zero at infinity, which is *constant* on bdy A and *constant* on bdy B, and which satisfies

$$\iint_{\text{bdy } A} \frac{\partial u}{\partial n} dS = Q > 0 \qquad \text{and} \qquad \iint_{\text{bdy } B} \frac{\partial u}{\partial n} dS = 0.$$

[*Interpretation*: The harmonic function $u(\mathbf{x})$ is the electrostatic potential of two conductors, A and B; Q is the charge on A, while B is uncharged.]

(a) Show that the solution is unique. (*Hint:* Use the Hopf maximum principle.)

(b) Show that $u \geq 0$ in D. [*Hint:* If not, then $u(\mathbf{x})$ has a negative minimum. Use the Hopf principle again.]

(c) Show that $u > 0$ in D.

Exercise 7.2

1. Derive the representation formula for harmonic functions (7.2.5) in two dimensions.

2. Let $\phi(\mathbf{x})$ be any C^2 function defined on all of three-dimensional space that vanishes outside some sphere. Show that

$$\phi(0) = -\iiint \frac{1}{|\mathbf{x}|}\Delta\phi(\mathbf{x})\frac{d\mathbf{x}}{4\pi}.$$

The integration is taken over the region where $\phi(x)$ is not zero.

3. Give yet another derivation of the mean value property in three-dimensions by choosing D to be a ball and x_0 its center in the representation formula (1).

Exercise 7.3

1. Show that the Green's function is unique. (*Hint:* Take the difference of two of them.)

3. Verify the limit of A_ϵ as claimed in the proof of the symmetry of the Green's function.

Exercise 7.4

1. Find the one-dimensional Green's function for the interval $(0, l)$. The three properties defining it can be restated as follows.

   (i) It solves $G''(x) = 0$ for $x \neq x_0$ ("harmonic").

   (ii) $G(0) = G(l) = 0$.

   (iii) $G(x)$ is continuous at x_0 and $G(x) + \frac{1}{2}|x - x_0|$ is harmonic at x_0.

2. Verify directly from (3) or (4) that the solution of the half-space problem satisfies the condition at infinity:

$$u(\mathbf{x}) \to 0 \qquad \text{as } |\mathbf{x}| \to \infty.$$

   Assume that $h(x, y)$ is a continuous function that vanishes outside some circle.

3. Show directly from (3) that the boundary condition is satisfied: $u(x_0, y_0, z_0) \to h(x_0, y_0)$ as $z_0 \to 0$. Assume $h(x, y)$ is continuous and bounded. [*Hint:* Change variables $s^2 = [(x - x_0)^2 + (y - y_0)^2]/z_0^2$ and use the fact that $\int_0^\infty s(s^2 + 1)^{-3/2} ds = 1$.]

5. Notice that the function xy is harmonic in the half-plane $\{y > 0\}$ and vanishes on the boundary line $\{y = 0\}$. The function 0 has the same properties. Does this mean that the solution is not unique? Explain.

6. (a) Find the Green's function for the half-plane $\{(x, y) : y > 0\}$.

   (b) Use it to solve the Dirichlet problem in the half-plane with boundary values $h(x)$.

   (c) Calculate the solution with $u(x, 0) = 1$.

7. (a) If $u(x, y) = f(x/y)$ is a harmonic function, solve the ODE satisfied by f.

   (b) Show that $\partial u/\partial r \equiv 0$, where $r = \sqrt{x^2 + y^2}$ as usual.

   (c) Suppose that $v(x, y)$ is any function in $\{y > 0\}$ such that $\partial v/\partial r \equiv 0$. Show that $v(x, y)$ is a function of the quotient x/y.

   (d) Find the boundary values $\lim_{y \to 0} u(x, y) = h(x)$.

   (e) Show that your answer to parts (c) and (d) agrees with the general formula from Exercise 6.

9. Find the Green's function for the tilted half-space $\{(x, y, z) : ax + by + cz > 0\}$. (*Hint:* Either do it from scratch by reflecting across the tilted plane, or change variables in the double integral (3) using a linear transformation.)

10. Verify the formula (11) for $G(\mathbf{x}, 0)$, the Greens function with its second argument at the center of the sphere.

13. Find the Green's function for the tilted half-ball $D = \{x^2 + y^2 + z^2 < a^2,\ z > 0\}$. (*Hint:* The easiest method is to use the solution for the whole ball and reflect it across the plane.)

15. (a) Show that if $v(x, y)$ is harmonic, so is $u(x, y) = v(x^2 - y^2, 2xy)$.

    (b) Show that the transformation $(x, y) \mapsto (x^2 - y^2, 2xy)$ maps the first quadrant onto the half-plane $\{y > 0\}$. (*Hint:* Use either polar coordinates or complex variables.)

17. (a) Find the Green's function for the quadrant

$$Q = \{(x, y) : x > 0, y > 0\}.$$

    (*Hint:* Either use the method of reflection or reduce to the half-plane problem by the transformation in Exercise 15.)

19. Consider the four-dimensional laplacian $\Delta u = u_{xx} + u_{yy} + u_{zz} + u_{ww}$. Show that its fundamental solution is $r^{-\frac{3}{2}}$, where $r^2 = x^2 + y^2 + z^2 + w^2$.

20. Use Exercise 19 to find the Green's function for the half-hyperspace $\{(x, y, z, w) : w > 0\}$.

21. The *Neumann function* $N(x, y)$ for a domain D is defined exactly like the Green's function in Section 7.3 except that (ii) is replaced by the Neumann boundary condition

(ii)* $$\frac{\partial N}{\partial n} = 0 \quad \text{for } x \in \text{ bdy } D.$$

    State and prove the analog of Theorem 7.3.1, expressing the solution of the Neumann problem in terms of the Neumann function.