Mathematical Analysis III

Tutorial 5 (October 24)

The following were discussed in the tutorial this week:

1. Show that every open set in $\mathbb{R}$ can be written as a countable disjoint union of open intervals.

2. Let $\Omega = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc on complex plane. Let $C(\Omega)$ be the set of all $\mathbb{C}$-valued continuous functions on Ω.

   (a) For each $m \geq 1$, let

   $$d_m(f,g) = \sup_{|z|\leq 1-1/m} |f(z) - g(z)|, \quad \text{for all } f, g \in C(\Omega).$$

   Define

   $$d(f,g) = \sum_{m=1}^{\infty} 2^{-m} \frac{d_m(f,g)}{1 + d_m(f,g)}, \quad \text{for all } f, g \in C(\Omega).$$

   Show that d is a metric on $C(\Omega)$.

   (b) Given $f_n, f \in C(\Omega)$, we say that f_n converges to f uniformly on compact sets if for every compact (i.e. closed and bounded in this case) $K \subseteq \Omega$, $f_n|_K$ converges to $f|_K$ uniformly. Show that f_n converges to f uniformly on compact sets if and only if $d(f_n, f) \to 0$ as $n \to \infty$.

3. Recall the definition of *compactness*, *sequential compactness* and their equivalence in a metric space.

4. Let (X, d_1), (Y, d_2) be two metric spaces. Let $(X \times Y, d)$ be the product space endowed with the metric

   $$d((x_1, y_1), (x_2, y_2)) := \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}.$$

   If $A \subset X$ and $B \subset Y$ are compact, show that $A \times B$ is compact in $X \times Y$.

5. Let X, Y be two metric spaces. Let $f : X \to Y$ be a continuous bijection. If X is compact, show that f^{-1} is also continuous.