Mathematical Analysis III

Tutorial 4 (October 17)

The following were discussed in the tutorial this week:

Let (X, d) be a metric space. $E \subset X$.

1. Recall the notions of boundary, closure and interior of sets in a metric space.
2. We prove the following properties of interior as stated in the lecture notes:
   (i) E° is open.
   (ii) $E^\circ = E \setminus \partial E$.
   (iii) $E^\circ = X \setminus (\overline{X \setminus E})$.
   (iv) $E^\circ = \bigcup\{G : G \text{ is an open set }, G \subset E\}$.
3. Let $A, B \subset X$. Show that $(A \cap B)^\circ = A^\circ \cap B^\circ$. Is it true that $(A \cup B)^\circ = A^\circ \cup B^\circ$? How about infinite intersection?
4. Suppose $E \neq \emptyset$, recall that $\rho_E : X \to \mathbb{R}$ is a continuous function defined by
   $$\rho_E(x) = \inf_{y \in E} d(x, y) \quad \text{for } x \in X.$$
   Show that
   (a) if $E \neq \emptyset$, then $\overline{E} = \{x \in X : \rho_E(x) = 0\}$;
   (b) if $E \neq X$, then $E^\circ = \{x \in X : \rho_{X \setminus E}(x) > 0\}$.
5. Write
   $$B_r(x) := \{y \in X : d(x, y) < r\} \quad \text{and} \quad C_r(x) := \{y \in X : d(x, y) \leq r\}.$$
   Show that $\overline{B_r(x)} \subset C_r(x)$ for any $x \in X$, $r > 0$. Is it true that $\overline{B_r(x)} = C_r(x)$? What if the metric space (X, d) is replaced by a normed vector space $(X, \|\cdot\|)$?
6. Show that
   $$F := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } x^2 \leq y \leq x\}$$
   is closed in $\mathbb{R}^2$. (**Hint:** Consider the continuous functions $f(x, y) = x$, $g(x, y) = y - x^2$, $h(x, y) = x - y$.)