Solution 1

1. Let f be a 2π-periodic function integrable on $[-\pi, \pi]$. Show that it is integrable over any finite interval and
$$\int_I f(x)dx = \int_{-\pi}^{\pi} f(x)dx,$$
for any interval I of length 2π.

Solution It is clear that f is also integrable on $[n\pi, (n+2)\pi]$, $n \in \mathbb{Z}$, so it is integrable on any finite interval. Let $I = [a, a+2\pi]$ for some real number a. Since the length of I is 2π, there exists some n such that $n\pi \in I$ but $(n+2)\pi$ does not belong to the interior of I. We have
$$\int_a^{a+2\pi} f(x)dx = \int_a^{n\pi} f(x)dx + \int_{n\pi}^{a+2\pi} f(x)dx.$$
Using
$$\int_a^{n\pi} f(x)dx = \int_{a+2\pi}^{(n+2)\pi} f(x)dx$$
(by a change of variables), we get
$$\int_a^{a+2\pi} f(x)dx = \int_{a+2\pi}^{(n+2)\pi} f(x)dx + \int_{n\pi}^{a+2\pi} f(x)dx = \int_{n\pi}^{(n+2)\pi}.$$
Now, using a change of variables again we get
$$\int_{n\pi}^{(n+2)\pi} f(x)dx = \int_{-\pi}^{\pi} f(x)dx.$$

2. Show that the Fourier series of every even function is a cosine series and the Fourier series of every odd function is a sine series.

Solution Write
$$f(x) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$
Suppose $f(x)$ is an even function. Then, for $n \geq 1$, we have
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx f(x)dx = \frac{1}{\pi}[\int_{-\pi}^{0} \sin nx f(x)dx + \int_0^{\pi} \sin nx f(x)dx].$$
By a change of variable and using $f(-x) = f(x)$ since $f(x)$ is an even function,
$$\int_{-\pi}^{0} \sin nx f(x)dx = \int_0^{\pi} \sin(-nx)f(-x)dx = -\int_0^{\pi} \sin nx f(x)dx,$$
one has
$$b_n = \frac{1}{\pi}[-\int_0^{\pi} \sin nx f(x)dx + \int_0^{\pi} \sin nx f(x)dx] = 0.$$
Hence the Fourier series of every even function f is a cosine series.

Now suppose $f(x)$ is an odd function. Then, for $n \geq 1$, we have
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos nx f(x)dx = \frac{1}{\pi}[\int_{-\pi}^{0} \cos nx f(x)dx + \int_0^{\pi} \cos nx f(x)dx].$$

By a change of variable and using $f(-x) = -f(x)$ since $f(x)$ is an odd function,

$$\int_{-\pi}^{0} \cos nx f(x)dx = \int_{0}^{\pi} \cos(-nx)f(-x)dx = -\int_{0}^{\pi} \cos nx f(x)dx,$$

one has

$$a_n = \frac{1}{\pi}[-\int_0^\pi \cos nx f(x)dx + \int_0^\pi \cos nx f(x)dx] = 0.$$

Furthermore, by a change of variable and using $f(-x) = -f(x)$,

$$\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}[\int_{-\pi}^{0} f(x)dx + \int_0^\pi f(x)dx] \\ &= \frac{1}{2\pi}[-\int_0^\pi f(x)dx + \int_0^\pi f(x)dx] = 0. \end{aligned}$$

Hence the Fourier series of every odd function f is a sine series.

3. Each of the following functions (on the left hand side) are defined on $[-\pi, \pi]$. Sketch the 2π-periodic expansion and verify their Fourier expansion on the right hand side.

   (a)
   $$x^2 \sim \frac{\pi^2}{3} - 4\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx,$$

   (b)
   $$|x| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x,$$

   (c)
   $$f(x) = \begin{cases} 1, & x \in [0, \pi] \\ -1, & x \in [-\pi, 0] \end{cases} \sim \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x,$$

   (d)
   $$g(x) = \begin{cases} x(\pi - x), & x \in [0, \pi) \\ x(\pi + x), & x \in (-\pi, 0) \end{cases} \sim \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin(2n-1)x.$$

Solution

(a) Consider the function $f_1(x) = x^2$. As $f_1(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi}\frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{\pi^2}{3},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx dx \\ &= \frac{1}{n\pi}x^2 \sin nx\Big|_{-\pi}^{\pi} - \frac{2}{n\pi}\int_{-\pi}^{\pi} x \sin nx dx \\ &= \frac{2}{n^2\pi}x \cos nx\Big|_{-\pi}^{\pi} - \frac{2}{n^2\pi}\int_{-\pi}^{\pi} \cos nx dx \\ &= 4\frac{(-1)^n}{n^2}. \end{aligned}$$

(b) Consider the function $f_2(x) = |x|$. As $f_2(x)$ is even, its Fourier series is a cosine series and hence $b_n = 0$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x|dx = \frac{1}{2\pi}\frac{x^2}{2}\bigg|_{-\pi}^{\pi} = \frac{\pi}{2},$$

and by integration by parts,

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\cos nx dx = \frac{2}{\pi}\int_0^{\pi} x\cos nx dx \\ &= \frac{2}{n\pi}x\sin nx\bigg|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin nx dx \\ &= -\frac{2}{n^2\pi}\cos nx\bigg|_0^{\pi} \\ &= -2\frac{[(-1)^n - 1]}{n^2\pi}. \end{aligned}$$

(c) As $f(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$.

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi}\sin nx dx \\ &= \frac{2}{n\pi}\cos nx\bigg|_0^{\pi} \\ &= 2\frac{[(-1)^n - 1]}{n\pi}. \end{aligned}$$

(d) As $g(x)$ is odd, its Fourier series is a sine series and hence $a_n = 0$. By integration by parts,

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx dx \\ &= -\frac{2}{n\pi}x(\pi - x)\cos nx\bigg|_0^{\pi} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos nx dx \\ &= \frac{2}{n^2\pi}(\pi - 2x)\sin nx\bigg|_0^{\pi} + \frac{4}{n^2\pi}\int_0^{\pi}\sin nx dx \\ &= -\frac{4}{n^3\pi}\cos nx\bigg|_0^{\pi} \\ &= -\frac{4}{n^3\pi}[(-1)^n - 1]. \end{aligned}$$

4. Consider the function $f(x) = x^2$ on $(0,2\pi]$ and its 2π-periodic extension $\tilde{f}$ by $\tilde{f}(x) = f(x - 2k\pi)$ for $x \in (2k\pi, 2(k+1)\pi]$, Sketch $\tilde{f}$ and show that

$$x^2 \sim \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos nx}{n^2} - 4\pi\sum_{n=1}^{\infty}\frac{\sin nx}{n},$$

for $x \in [0, 2\pi]$.

Solution

Consider the function $f(x) = x^2$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{2\pi}\frac{x^3}{3}\bigg|_0^{2} = \frac{4\pi^2}{3},$$

and by integration by parts,

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx dx \\
&= \frac{1}{n\pi}x^2 \sin nx\Big|_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi} x \sin nx dx \\
&= \frac{2}{n^2\pi}x \cos nx\Big|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \cos nx dx \\
&= \frac{4}{n^2},
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx dx \\
&= -\frac{1}{n\pi}x^2 \cos nx\Big|_0^{2\pi} + \frac{2}{n\pi}\int_0^{2\pi} x \cos nx dx \\
&= -\frac{4\pi}{n} + \frac{2}{n^2\pi}x \sin nx\Big|_0^{2\pi} - \frac{2}{n^2\pi}\int_0^{2\pi} \sin nx dx \\
&= -\frac{4\pi}{n}.
\end{aligned}$$